A Random-Walk or Color-Chaos on the Stock Market? - Time-Frequency Analysis of S&P Indexes

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Abstract

The random-walk (white-noise) model and the harmonic model are two polar models in linear systems. A model in between is color chaos which generates irregular oscillations with a narrow frequency (color) band. Time-frequency analysis is introduced for evolutionary time series analysis. The deterministic component from noisy data can be recovered by a time-variant filter in Gabor space. The characteristic frequency is calculated from the Wigner decomposed distribution series. It is found that about seventy percent of fluctuations in Standard & Poor stock price indexes, such as the FSPCOM and FSDXP monthly series, detrended by the Hodrick-Prescott (HP) filter, can be explained by deterministic color chaos. The characteristic period of persistent cycles is around three to four years. Their correlation dimension is about 2.5.

The existence of persistent chaotic cycles reveals new perspective of market resilience and new sources of economic uncertainties. The nonlinear pattern in the stock market may not be wiped out by market competition under nonequilibrium situations with trend evolution and frequency shifts. The color-chaos model of stock-market movements may establish a link between business-cycle theory and asset-pricing theory.

I. Introduction

Finance theory in equilibrium economics is based on the random-walk model of stock prices. However, there is a more general scenario: a mixed process with random noise and deterministic pattern, including a possibility of deterministic chaos.

Chaos is widely found in the fields of physics, chemistry, and biology. But the existence of economic chaos is still an open issue [Barnett and Chen 1988, Brock and Sayers 1989, Ramsey, Sayers, and Rothman 1990, DeCoster and Mitchell 1991, 1994, Barnett et. al. 1995]. Trends, noise, and time evolution caused by structural changes are the main difficulties in economic time series analysis. A more generalized spectral analysis is needed for testing economic chaos [Chen 1988, 1993].

Measurement cannot be separated from theory. There are two polar models in linear dynamics: white noise and harmonic cycle. Correlation analysis and spectral analysis are complementary tools in the stationary time-series analysis. White noise has a zero correlation and a flat spectrum while a harmonic cycle has an infinite correlation and a sharp spectrum with zero-width. Obviously, real data fall between these two extremes.

A major challenge in economic time series analysis is how to deal with time evolution. Econometric models, such as the ARCH and GARCH models with a changing mean and variance are parametric models in the nonstationary stochastic approach [Engle 1982, Bollerslev 1986]. A generalized spectral approach is more useful in studies of deterministic chaos [Chen 1993].

It is known that a stationary stochastic process does not have a stationary or continuous instantaneous frequency in time-frequency representation. Therefore, we do not use the terms stationary and nonstationary which are familiar in a stochastic approach. A new representation will introduce some conceptual changes. There are many fundamental differences between a nonlinear deterministic approach and a linear stochastic approach including time scales, observation references, and testing philosophy.

From the view of theoretical studies, the discrete-time white chaos generated by nonlinear difference equations is tractable in analytic mathematics and compatible with the optimization rationality [Day and Benhabib 1980, Benhabib 1990]. From the needs of empirical analysis, the continuous-time color chaos generated by nonlinear differential equations is more capable of describing business cycles than white chaos, because their erratic fluctuations and recurrent pattern can be characterized by nonlinear oscillations with irregular amplitude and a narrow frequency (color) band in spectrum [Chen 1988, 1993; Zarnowitz 1993].

We introduce the time-frequency representation as a non-parametric approach of generalized spectral analysis for the evolutionary time series [Qian and Chen 1996]. The Wigner distribution in quantum mechanics and the Gabor representation in communication theory were pioneered by two Nobel laureate physicists [Wigner 1933, Gabor 1948]. Applied scientists in signal processing have made fundamental progress in developing efficient algorithms of time-frequency distribution series [Qian and Chen 1994 a,b, 1996]. These are powerful tools in our studies of economic chaos [Chen 1994,1995,1996].

In dealing with problems of growing trends and strong noise, we apply the Hodrick-Prescott (HP) filter for trend-cycle decomposition [Hodrick and Prescott 1981] and time-variant filters in

Gabor space for pattern recognition [Qian and Chen 1996, Sun et. al. 1996]. We got clear signals of low-dimensional color chaos from Standard & Poor stock market indicators. The chaos signals can explain about 70 percent of stock variances from detrended cycles. Its characteristic period is around three to four years. Their correlation dimension is about 2.5. The time paths of their characteristic period is useful in analyzing cause and effect from historical events. Clearly, the color-chaos model describes more features of market movements than the popular random-walk model.

The newly decoded deterministic signals from persistent business cycles reveal new sources of market uncertainty and develop new ways of economic diagnostics and risk analysis. Friedman's argument against irrational speculators ignore the issue of information ambiguity in evolving economy and financial risk for rational arbitrageurs [Friedman 1953]. A nonlinear pattern in the stock market may not be wiped out by market competition because complexity and diversity in market behavior are generated by changing uncertainty, nonlinear overshooting, and time-delays in learning and feedback mechanism.

II. Roles of Time-Scale and Reference-Trend in Representation of Business Cycles

A distinctive problem in economic analysis is how to deal with growing trends in an aggregate economic time series. Unlike laboratory experiments in natural sciences, there is no way to maintain steady flows in economic growth and describe raw business-cycles by invariant attractors. Many controversial issues in macroeconomic studies, such as noise versus chaos in business cycles, are closely related to competing detrending methods [Chen 1988, 1993, Ramsey, Sayers and Rothman 1990, Brock and Sayers 1988].

The first issue is the time scale in economic representation. A continuous-time representation in the form of $[X(t), dX(t)/dt, \ldots, d^nX(t)/dt^n n]$ is widely used in science and engineering. It is an empirical question whether the dynamical system can be well approximated by a low-order vector up to the n-th order of derivatives. In Hamiltonian mechanics, n is 1 for mechanical systems because its future movement can be determined by the Newton's law of motion in addition to initial conditions in position and momentum. It means that both level (position) and rate (velocity) information are important in characterizing the underlying dynamical system. Chaos theory in nonlinear dynamics further emphasizes the role of history because a nonlinear deterministic system is sensitive to its initial conditions. In business-cycle studies, there is no consensus on the order n. The martingale theory of the stock market simply ignores the path-dependent information in the stock market. We will demonstrate that both level and rate information are important when correlations are not short during business cycles.

Econometricians often use differences in the form of $[X(t), \Delta X(t), \ldots, \Delta^m X(t)]$ in parametric modeling. We should note that these two representations are not equivalent. Mathematically, a one-dimensional differential equation $dX(t)/dt = F[x, t]$ can be approximated by m-th order difference equations. Numerically, m should be larger than one hundred when the numerical error is required to be less than one percent. Many econometricians favor the discrete-time difference equations instead of the continuous-time differential equations because of its mathematical convenience in regression analysis. However, a discrete-time representation is a two-order lower approximation of a similar continuous-time system.

The issue of choosing an appropriate time-sampling rate is often ignored in econometric analysis. Chaotic cycles in continuous-time may look like random if the sampling time-interval is not small compared to its fundamental period of the cycles. This issue is important in pattern recognition. For example, annual economic data are not capable of revealing the frequency pattern of business cycles. Numerically, a large time unit such as the annual time-series can easily obscure a cyclic pattern in correlation analysis of business cycles.

A related issue is how to choose a reference-trend or a proper transformation to simplify the empirical pattern of business cycles. Suppose, a new vector [G(t), C(t)] is defined in terms of the original vector [X(t), dX(t) / dt]. If C(t) is a bounded time series, then C(t) has a chance to be described by a deterministic attractor, or a stationary stochastic process. In business-cycle studies, finding a proper transformation is called the problem of trend-cycle decomposition or simply detrending. In astronomy, the critical trend-cycle problem was solved by Copernicus and Kepler by using a heliocentric reference system. In econometrics, the choice of observation reference is an open issue in business cycle theory [Zarnowitz 1992].

The core problem in economic analysis is not noise-smoothing but trend-defining in economic observation and decision making. A short-time deviation may be important for speculative arbitrageurs while the shape of the long-term trend can be critical to strategic investors. Certainly, investors in a real economy have diversified strategies and time-horizons. The interactive nature of social behavior often form some consensus on business cycles. This fact suggests that a relative preferred reference exists in economic studies. We will show that the HP filter in trend-cycle decomposition is a promising way to define the growth trend in business cycles.

It is the theoretical perspective which dictates the choice of a detrending approach. Econometric practice of pre-whitening data is justified by equilibrium theory and convenient for regression analysis. For example, a Frisch-type noise-driven model of business cycles will end with white noise after several damped oscillations [Frisch 1933]. For pattern recognition, a typical technique in science and engineering is to project the data on some well-constructed deterministic space to recover possible patterns from empirical time series. Notable examples are the Fourier analysis and wavelets.

There are two criteria in choosing the proper mathematical representation: mathematical reliability and empirical verifiability. Unlike experimental economics, macroeconomic time series are not reproducible in history. Traditional tests in econometric analysis have limited power in studies of an evolutionary economy containing deterministic components. For example, testing the whiteness of residuals or comparing mean squared errors have little power when the real economy is not a stationary stochastic process. A good fit of past data does not guarantee the ability for better future predictions. The outcome of out of sample tests in a simulation experiment depends on the choice of testing period, because structural changes vary in economic history.

To avoid the above problems in time-frequency analysis, we will use historical events as natural experiments to test our approach. Future laboratory experiments are possible in testing the martingale model and the color-chaos model in market behavior.

III. Trend-Cycle Decomposition and Time-Window in Observation

The linear detrending approach dominates econometric analysis because of its mathematical simplicity. There are two extreme approaches in econometric analysis: The trend-stationary (TS) approach of log-linear detrending (LLD) and the difference-stationary (DS) approach of first differencing (FD) [Nelson and Plosser 1982].

$$X_{FD}(t) = \log S(t) - \log S(t-1) = \log[\frac{S(t)}{S(t-1)}] \quad (3.1)$$

$$X_{LLDc}(t) = \log S(t) - (a + bt) \quad (3.2)$$

A compromise between these two extremes is the HP filter [Hodrick and Prescott 1981]. The HP smooth growth trend { HPs = G(t) } is obtained by minimizing the following function:

$$Min \sum [X(t) - G(t)]^2 + \lambda \sum \{[G(t+1) - G(t)] - [G(t) - G(t-1)]\}^2 \quad (3.3)$$

Here, λ is 1,600 for quarterly data and 14,400 for monthly, suggested by Kydland. LLD cycles of $X_{LLDc}(t)$ are residuals from log-linear trend. LLD trend can be considered as the limiting case of the HP trend when λ goes to infinity for logarithmic data.

In principle, a choice of observation reference is associated with a theory of economic dynamics. Log-linear detrending implies a constant exponential growth which is the base case in the neo-classical growth theory. The FD detrending produces a noisy picture that is predicted by the geometric random walk model with a constant drift (or the so-called unit-root model in econometric literature). The efficient market hypothesis simply asserts that stock price movement is a martingale which is unpredictable in finance theory.

Economically speaking, the FD detrending in econometrics implies that the level information in price indicators can be ignored in economic behavior. This assertion may conflict with many economic practices, because traders constantly watch economic trends, no one will make an investment decision based only on the current rate of price changes. Most economic contracts, including margin accounts in stock trading, are based on nominal terms. The error-correction model in econometrics tried to remedy the problem by adding some lagged-level information, such as using a one-year-before level as an approximation of the long-run equilibrium [Baba, Hendry, and Starr 1992]. Then comes the problem of what is the long-run equilibrium in the empirical sense. Option traders based on the Black-Scholes model find that it is extremely difficult to predict the mean, variance, and correlations from historical data [Merton 1990]. A proper decomposition of trend and cycles may find an appropriate scheme to weigh short-term and long-run impact of economic movements in economic dynamics.

From the view of complex systems, the linear approach is not capable of describing complex patterns of business cycles [Day and Chen 1993]. We need a better alternative of detrending. Statistically, a unit-root model can be better described by a nonlinear trend [Bierens 1995]. The question is what kind of trend is proper for catching the pertinent feature of the underlying mechanism. We can only solve the issue by comparing empirical information revealed from competing approaches.

The essence of trend-cycle decomposition is to find an appropriate time-window, or equivalently a proper frequency-window, in observing non-stationary movements. From the view of signal processing, log-linear detrending is a low-pass filter or wave-detector, while first differencing is a high-pass filter or noise-amplifier. Obviously, the FD filter is not helpful for detecting low frequency cycles.

Early evidence of economic chaos is found in TS detrended data [Barnett and Chen 1988, Chen 1988]. The main drawback of LLD detrending is its over-dependence of historical boundaries while the DS series is too erratic by amplifying high frequency noise [Friedman 1969]. The HP filter has two advantages. First, it is a localized approach in detrending without the problem of boundary-dependence. Second, its frequency response is in the range of business cycles [King and Rebelo 1993]. Some economists argue that the HP filter may transform a unit-root process into false cycles. A similar argument is also valid for the unit-root school, because the FD filter obscures complex cycles by amplifying random noise. No numerical experiment can solve a philosophical issue. In the history of science, the choice of a proper reference can only be solved as an empirical issue, i.e. whether we can discover some patterns and regularities that are relevant to economic reality. We will see that introducing a time-frequency representation and the HP filter does reveal some historical features of business cycles, that are not observable through the FD filter.

In this report, we will demonstrate tests of two monthly time series from the stock market indicators: FSPCOM is the Standard & Poor 500 stock price composite monthly index, and FSDXP, the S&P common stock composite dividend yield. The source of the data is the Citibase. The data covers a period from 1947 to 1992. To save space, we only give the plots from the FSPCOM data. More tests in macroeconomic aggregates are reported elsewhere [Chen 1994, 1995, 1996].

The role of detrending in shaping characteristic statistics can be seen in Table I. For most economic time series, the magnitude of variance (a key parameter in asset pricing theory) and the length in autocorrelations (a key parameter in statistical tests) are closely associated with the characteristic time-window of the underlying detrending method. The variance observed by HP detrending is about 5.7 times of that of FD, while the HP decorrelation length is 4.6 times of FD. Their ratio in variance is roughly in the same order as the ratio in the decorrelation length.

Table 1

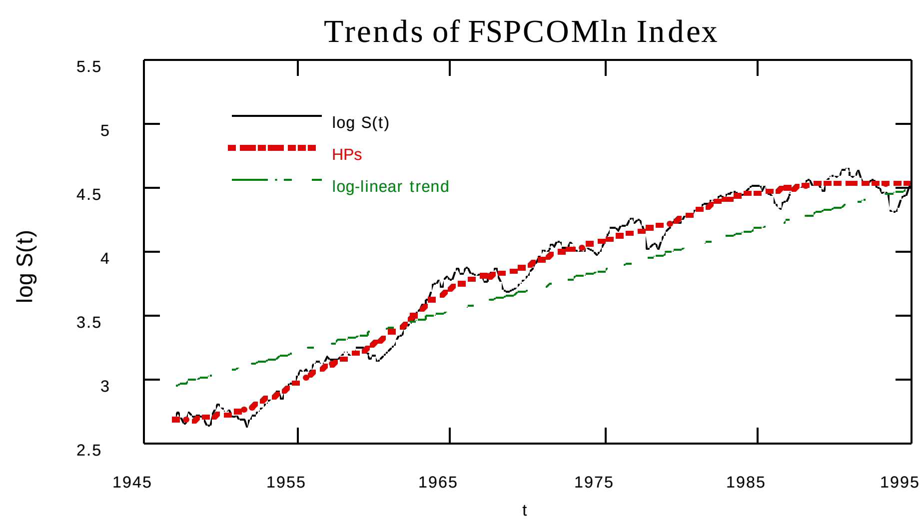
Table I. Detrending statistics for FSPCOMln monthly

Detrending	Mean	STD	Variance	T_0 (month)	P_{dc} (year)

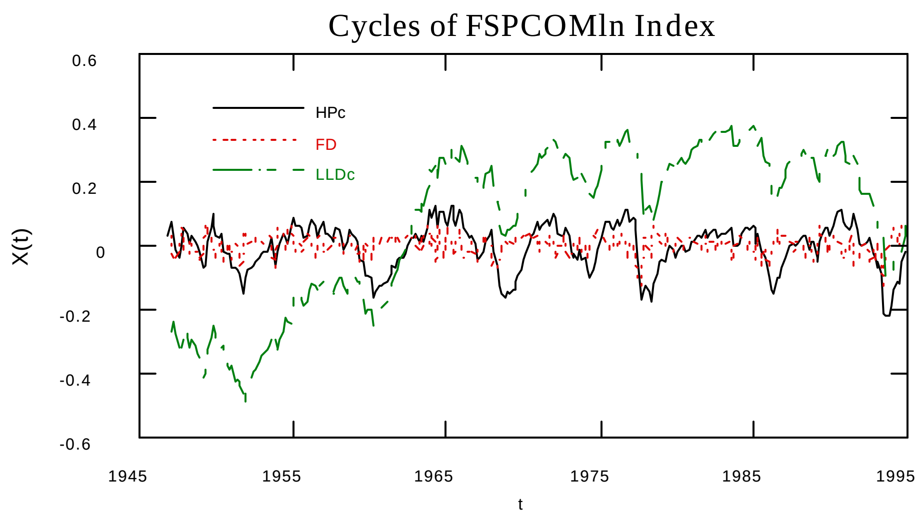
FD	0.012	0.1123	0.0126	1.94	0.7
HP	0.008	0.2686	0.0722	8.93	3.0
LLD	0.427	0.3265	0.1066	85.6	28.5

Here, T_0 is the decorrelation length measured by the time lag of the first zero in autocorrelations; P_{dc}, the decorrelation period for implicit cycles: $P_{dc} = 4T_0$.

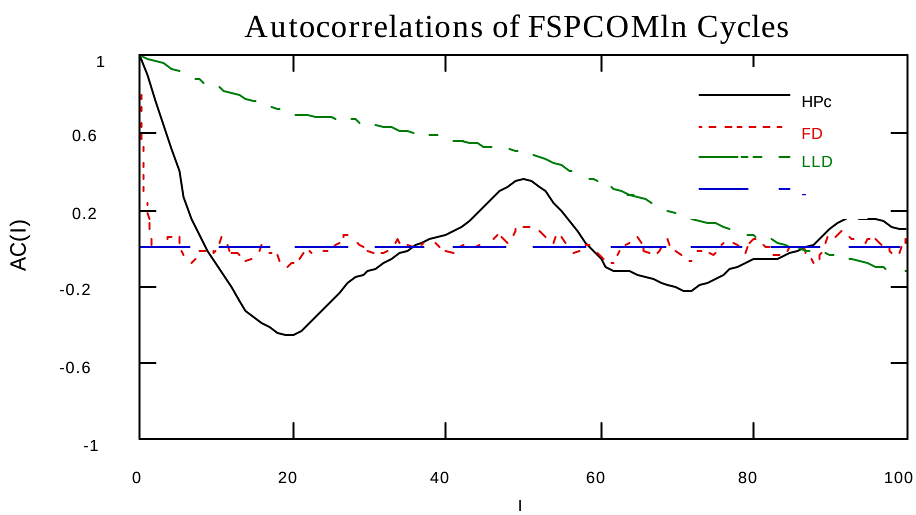
A typical example of an economic time series is shown in the logarithmic FSPCOM [see Fig. 1]. The contrast between the erratic feature of the DS series and the wavelike feature of TS and HP cycles is striking. For example, their lengths of autocorrelations are greatly varied. The autocorrelation length is the largest for LLD cycles, shortest for FD series, and in between for HP cycles.



(1a). HP trend and LLD (log-linear) trend for X(t) {=log S(t) }. LLDc cycles are residuals from log-linear trend.



(1b). Cycles from competing detrendings.



(1c). Autocorrelations of detrended series. The length of correlations varies for competing detrendings.

Fig. 1. Fluctuation patterns from competing trend-cycle decompositions, including FD, HP, and LLD detrending, for the logarithmic FSPCOM monthly series (1947-92). N=552.

IV. Instantaneous Auto-Correlations and Instantaneous Frequency in Time-Frequency Representation

In spectral representation, a plane wave has an infinite time-span but a zero-width in frequency domain. In a correlation representation, a pulse has a zero-width time-span but a full window in frequency space. To overcome their shortcomings, the wavelet representation with a finite span both in time and frequency (or scale) can be constructed for an evolutionary time series. The simplest time-frequency distribution is the short-time Fourier transform (STFT) by imposing a shifting finite time-window in the conventional Fourier spectrum.

The concepts of instantaneous auto-correlation and instantaneous frequency are important in developing generalized spectral analysis. A symmetric window in a localized time interval is introduced in the instantaneous autocorrelation function in the bilinear Wigner distribution (WD), the corresponding time-dependent frequency or simply time-frequency can be defined by the Fourier spectrum of its autocorrelations [Wigner1932]:

$$WD(t,\omega) = \int S(t+\frac{\tau}{2})\, S^*(t-\frac{\tau}{2}) \exp(-i\omega\tau)\, d\tau \qquad (4.1)$$

Continuous time-frequency representation can be approximated by a discretized two-dimensional time-frequency lattice. An important development in time-frequency analysis is the linear Gabor transform which maps the time series into the discretized two-dimensional time-frequency space [Gabor 1946]. According to the uncertainty principle in quantum mechanics and information theory, the minimum uncertainty only occurs for the Gaussian function.

$$\Delta t \ \Delta f \geq \frac{1}{4\pi} \qquad (4.2)$$

where Δt measures the time uncertainty, Δf the frequency uncertainty (angular frequency: $\omega = 2\pi f$).

Gabor introduced the Gaussian window in non-orthogonal base function h(t).

$$S(t) = \sum_{m,n} C_{m,n} \ h_{m,n}(t) \qquad (4.3)$$

$$h_{m,n}(t) = a^* \exp[-\frac{(t-m\Delta t)^2}{(2L)^2}]^* \exp(-i\, nt\, \Delta\omega) \qquad (4.4)$$

where Δt is the sample time-interval, Δw the sample frequency-interval, L the normalized Gaussian window-size, m and n the time and frequency coordinate in discretized time-frequency space [Qian and Chen 1994a].

The discrete-time realization of the continuous-time Wigner distribution can be carried out by the orthogonal-like Gabor expansion in discrete time and frequency [Qian and Chen 1994b, 1996][1] The time-frequency distribution series can be constructed as the decomposed Wigner distribution.

$$TFDS_D(t,w) = \sum_{0}^{D} P_d(t,w) \tag{4.5}$$

where P_d (t, ω) is the d-th order of decomposed Wigner distribution, d is measured by the maximum distance between interacting pairs of base functions. The zero-th order of a time-frequency distribution series without interferences leads to STFT. The infinite order converges to the Wigner distribution including higher interference terms. For an applied analysis, 2nd or 3rd order is a good compromise in characterizing frequency representation without severe cross-term interference. In our studies, we take the highest order D=3.

For comparison between the deterministic model and the stochastic model, we also demonstrate the time-frequency pattern of an AR(2) model of FD series.

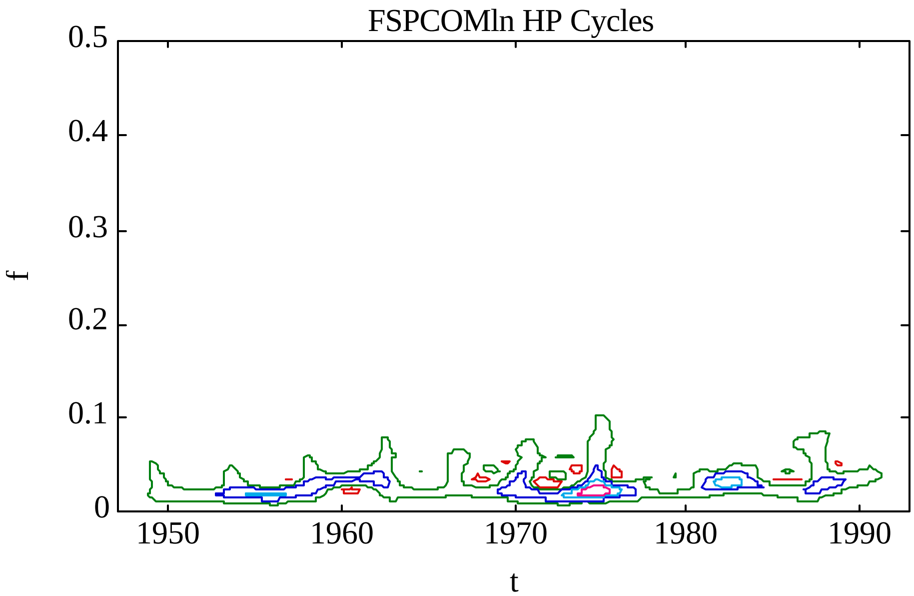
$$X(t) = 0.006\,[0.002] + 0.265\,[0.043]\,X(t-1) - 0.081\,[0.043]\,X(t-2) + m(t) \tag{4.6}$$

Here, standard deviations are in parenthesis, the residual μ(t) is white noise, its standard deviation is: $\sigma = 0.033$.

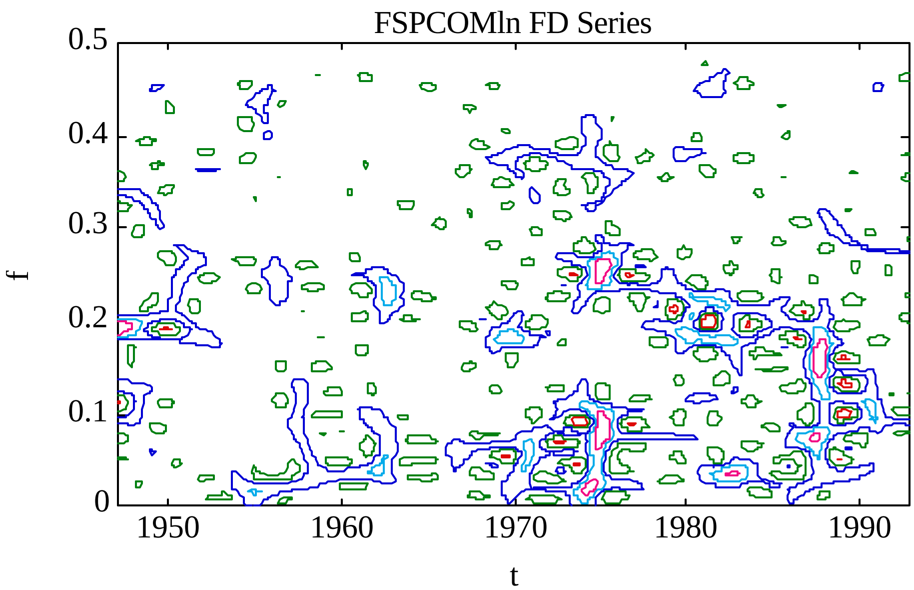
The deterministic cycle is characterized by a narrow horizontal frequency band in time-frequency space, while noise signals featured by drop-like images are evenly scattered in whole time-frequency space. We can see that FD series are very noisy while HP cycles have a clear trace of persistent cycles in the range of business cycle frequency. Later we will show that a stationary stochastic model, such as an AR(2) model of FD series, has a typical feature of color noise without a continuous frequency-line in time-frequency representation. A noise-driven model such as an AR or GARCH series can produce pseudo-cycles in Fourier spectrum, but cannot produce persistent cycles in time-frequency representation.

The time-frequency representations of the logarithmic FSPCOM HP cycles and FD series are shown in Fig. 2.

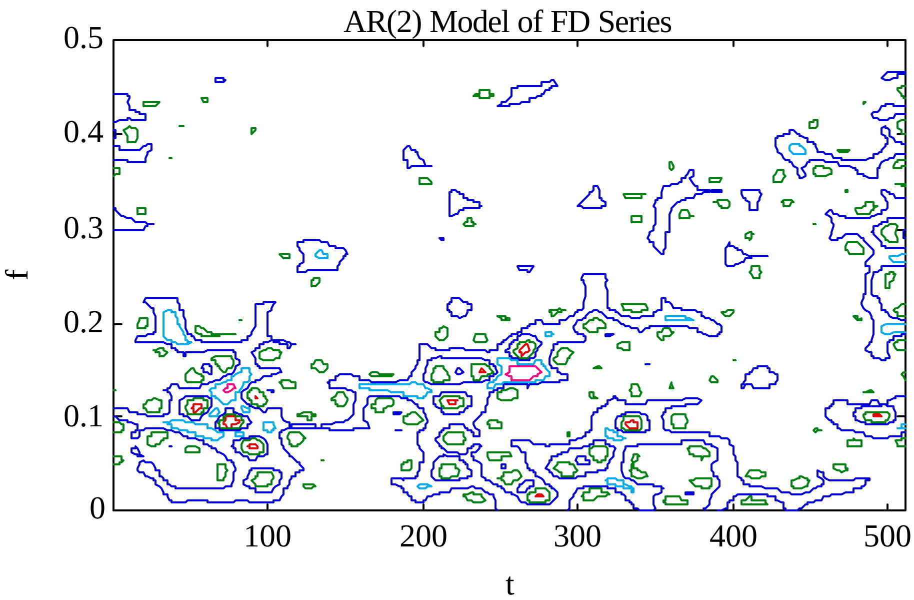
[1] The numerical algorithm is called the time-frequency distribution series (TFDS). The computer software is marketed by National Instruments under the commercial name of Gabor spectrogram as a tool kit in the Lab View System.



(2a). FSPCOMln HP cycles.



(2b). FSPCOMln FD series.



(2c). AR(2) model of FSPCOMln FD series. N=512.

Fig. 2. Time-frequency representation of empirical and simulated series from FSPCOMln detrended data. The X axis is the time t (or number of points); the Y axis is frequency f, the Z axis is the intensity of TFDS distribution.

For the deterministic mechanism, signal energy or variance is highly localized in time-frequency space. For example, the signal of FSPCOM HP cycles are concentrated in the lowest quarter frequency zone. Its characteristic period P_c is 3.9 years; 89 percent of its variance is concentrated within a bandwidth of a 12 percent frequency window, 73 percent within a 5 percent frequency window.

V. Time-Variant Filter in Gabor Space

The task of removing background noise is quite different in the trajectory representation and in time-frequency representation. It is very difficult to judge a good regression simply based on a residual test in econometrics. It is much easier to examine the linear Gabor distribution in the time-frequency space. We want to find a simple way to extract the main area with a high energy concentration, which can be reconstructed into a time series resembling main features of the original data. We will see if the filtered time series can be described by a simple deterministic oscillator.

For a stationary stochastic process, a linear filter can be applied. For an evolutionary process containing both deterministic and stochastic components, a time-variant nonlinear filter does a better job. The simplest time-variant filter is a mask function that mark the boundaries of the energy concentration area.

It is much easier to construct a time-variant filter based on the Gabor transform than on the Wigner transform, because the Gabor transform is linear. The original time series $X_o(t)$ can be represented by a M*N matrix in Gabor space. Its element C(m, n) has M points in the time frame and N points in the frequency frame. There is no absolute dividing line between cycles and noise in Gabor representation. We can define the thresholds of a peak distribution in frequency space at each time-section n. Correspondingly, the constructed mask operator Φ provides a simple time-varying filter that sets all outside Gabor coefficients to zero. To ensure the reconstruction is as close as possible to the ideal signal within the masked region in Gabor space, an iteration procedure is employed. After k-th iteration, we obtain the reconstructed time series Xg(t):

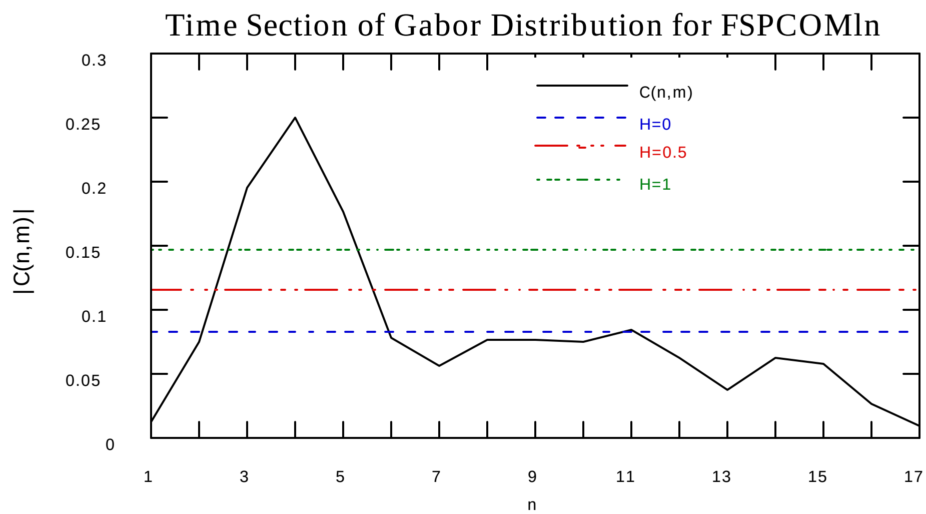
$$X_g(t) = \{\Gamma^{-1}\,\Phi\,\Gamma\}^k \; X(t) = \Theta^k \; X(t) \tag{5.1}$$

Where Γ and Γ^{-1} denote a forward and inverse Gabor transform in discrete time-frequency lattice space respectively. This process will converge as long as the maximum eigenvalue of the matrix $\Theta = \{\Gamma^{-1}\,\Phi\,\Gamma\}$ is less than one [Sun, Qian, et al 1995]. Our numerical calculation indicates that Θ converges in less than five iterations. The construction of the mask function in Gabor space is determined by the peak time-section of Gabor distribution [see Fig.3].

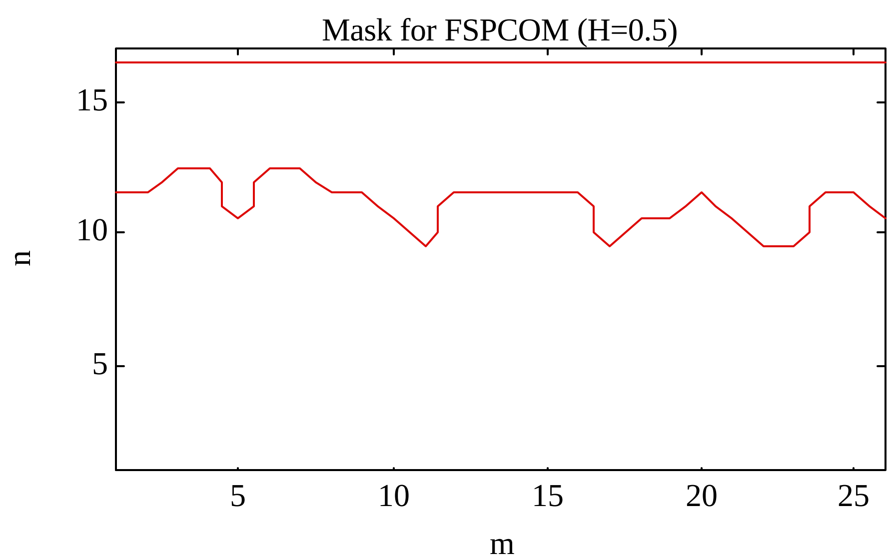
In order to define the boundaries of a time-varying filter, the cut-off threshold C_{th} at each time-section is introduced in the following way:

$$C_{th} = C_{mean} + H * C_{std} \tag{5.2}$$

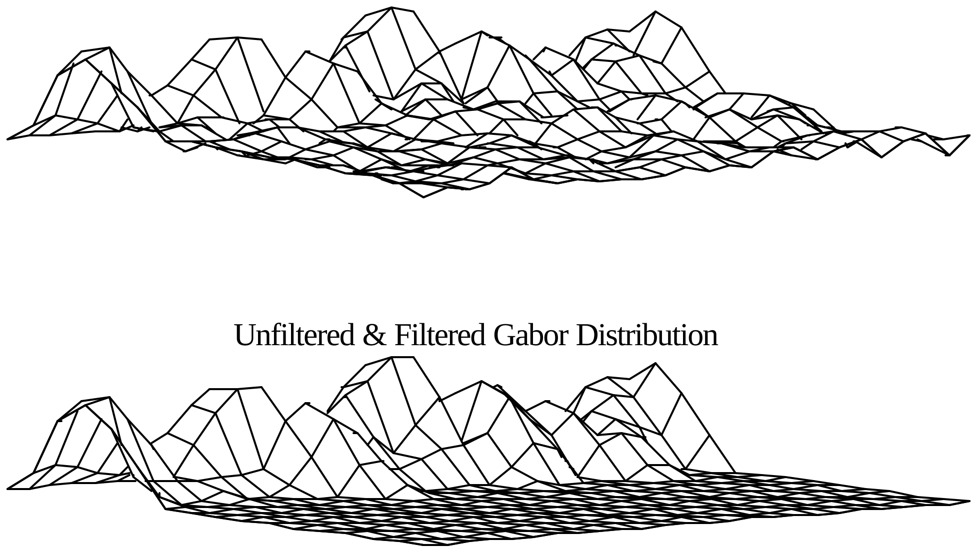
Here, H is the only adjustable parameter in setting the mask function. C_{mean} is the mean value of $|C(m,n)|$; C_{std}, the standard deviation of $|C(m,n)|$, all calculations are conducted at the peak time-section where $|C(m,n)|$ reaches the maximum value.



(3a). Peak time-section of Gabor distribution |C(m,n)| in the frequency domain for FSPCOMln. Different C_{th} values are indicated by different H. The X axis is the discrete number in frequency.



(3b). Mask function M(n, m) for FSPCOM. The tested time series is studied under the one fourth frequency window. For reducing the boundary distortion, the reflective boundaries at the both side of the data are added. The window size is the same as the Gaussian window of length L in Eqn.(3.2). So, the total length of data for Gabor transform is: N'=N/4+L*2. Here, N=552, L=64, the time sampling rate $\Delta\mu$=8, and C(n, m) is a matrix of 17*42. H=0.5.



(3c). The Gabor distribution for the unfiltered (upper) and filtered (lower) data.

Fig. 3. Construction and application of the time-variant filter in Gabor space. Unfiltered and filtered Gabor distribution for FSPCOMln HP cycles are demonstrated.

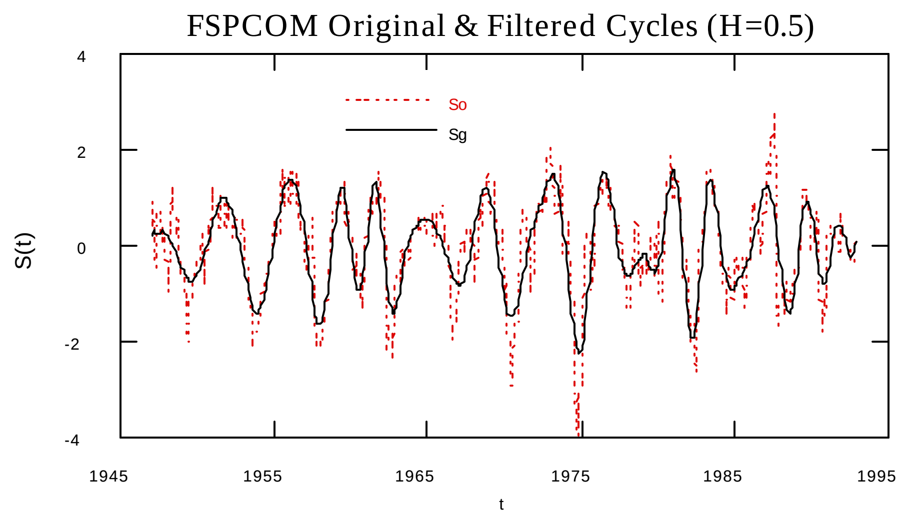
From Table II, we can see that the decomposition of variance is not sensitive to the choice of H, because the signal energy is highly concentrated in the low frequency band and the energy surface is very steep in the Gabor space. The variance of the filtered signal accounts for about 70 percent of total variance. We chose H=0.5 in later tests.

Table II. Decomposition of FSPCOMln data for varying H

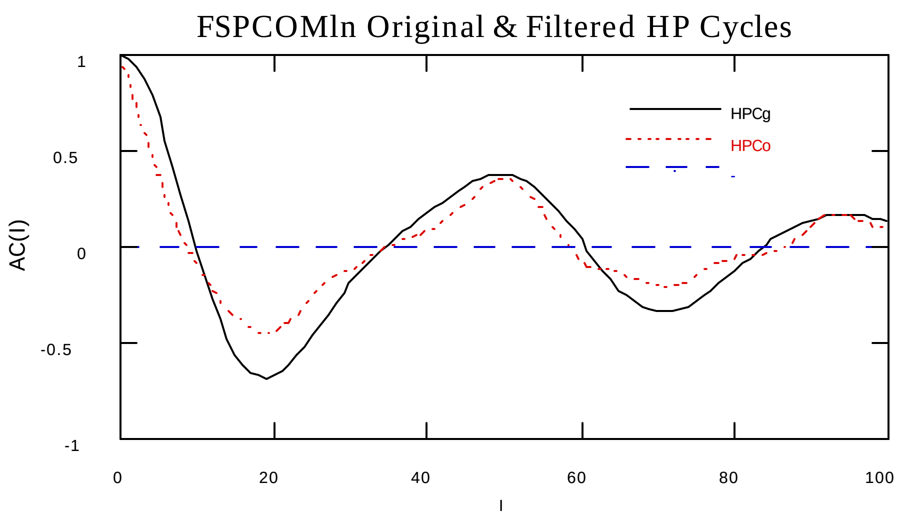
H	η	υ (%)	CCgo
0.0	0.8435	71.2	0.8595
0.5	0.8281	68.6	0.8471
1.0	0.8256	68.2	0.8461

The filtered HP cycles have clean features of a deterministic pattern while the filtered auto-regressive AR(2) series still has a random image [see Fig. 4]. Later we will see that the filtered HP cycles with a persistent frequency can be described by a color chaos with a low dimensionality.

Several statistics are calculated between the filtered and the original time series: η is the ratio of their standard deviation; υ, is the percentage ratio of variance; CCgo is their correlation coefficient.



(4a). The original and filtered FSPCOM HP cycles. η=82.8%; CCgo=0.847.



(4b). Autocorrelations of the original and reconstructed series. The time lag T_0 of the first zero in autocorrelations gives a rough measure of the cycle. The decorrelation Period: $P_{dc} = 4T_0$ =3.3 years (H=0.5).

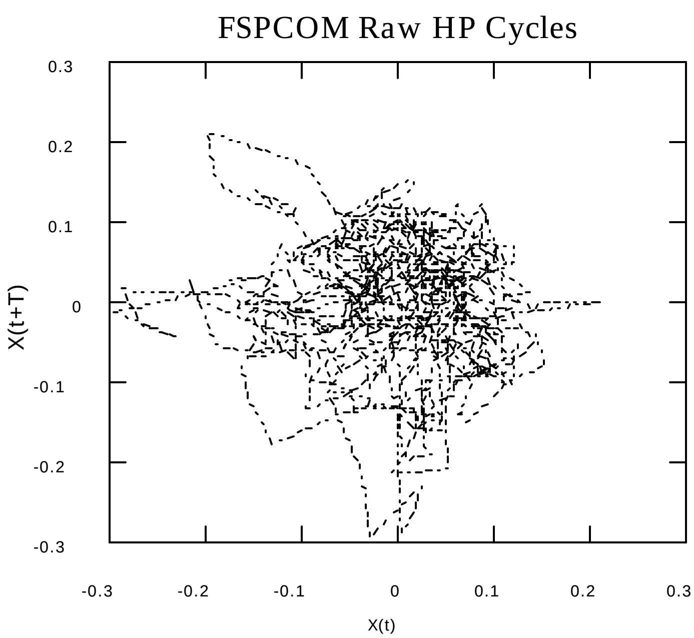
Fig. 4. The original and reconstructed time series of FSPCOMln HP Cycles.

The shape of the mask function is determined by the intensity of Gabor components. We should point out that a conventional test, such as the Durbin-Watson residual test, may not be applicable here, because residuals may be color noise. Our main target is catching the main deterministic pattern in the time-frequency space, not a parametric test based on regression analysis.

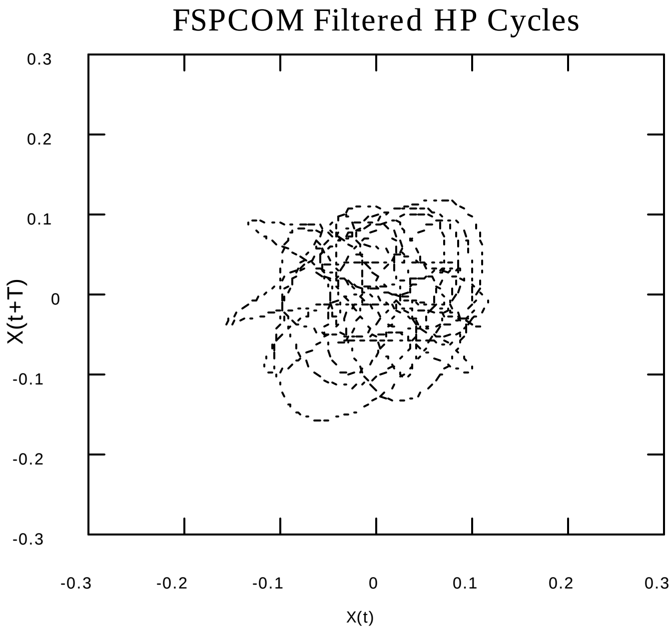
The reconstructed HPCg time series reveals the degree of deterministic approximation of business fluctuations: The correlation coefficient between the filtered and original series is 0.85.

Their ratio of standard deviation, η, is 85.8% for FSPCOM. In other words, about 73.7% of variance can be explained by a deterministic cycle with a well-defined characteristic frequency, even though its amplitude is irregular. This is a typical feature of chaotic oscillation in continuous-time nonlinear dynamical models.

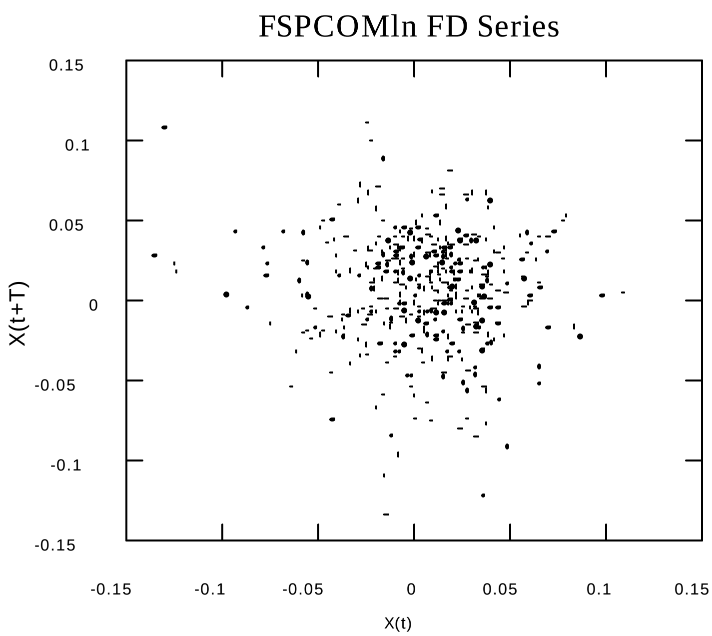
We can see that the phase portrait of filtered FSPCOMln HP cycles has a clear pattern of chaotic attractors, while the filtered AR(2) model fitting FSPCOMln FD series still keeps its random image [Fig.5].



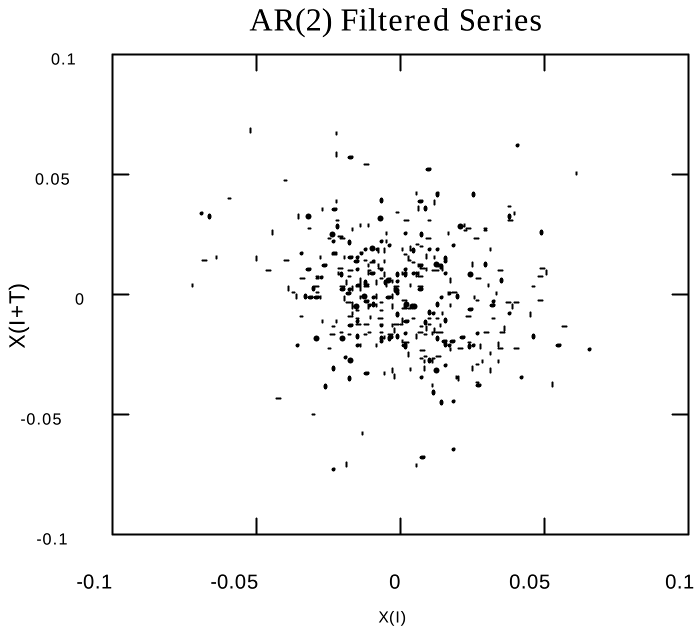
(5a). FSPCOMln HPc unfiltered series. T=60. Some pattern is emerged behind a noisy background.



(5b). FSPCOMln HPc filtered series. T=60. Clear pattern of strange attractor can be observed.



(5c). FSPCOMln FD series. T=40. The cloud-like pattern indicates the dominance of high frequency noise.



(5d). Filtered AR(2) series. T=5. No deterministic structure can be identified.

Fig. 5. Patterns of phase portraits for FSPCOMln series.

From Fig. 5, we also confirm our previous discussion in section 2 that FD detrending simply amplifies high frequency noise while HP detrending plus the time-variant filter in Gabor space pick up deterministic signals of color chaos from noisy data.

VI. Characteristic Frequency and Color Chaos

Time frequency representation contains rich information of underlying dynamics. At each section of time t, the location of the peak frequency f(t) can be easily identified from the peak of energy distribution in the frequency domain. If the time path of f(t) forms a continuous trajectory, we can define a characteristic frequency f_c from the time series. Correspondingly, we have a characteristic period P_c (= $1/f_c$). Stochastic time series such as the auto-regressive (AR) process cannot form a continuous line in time-frequency representation.

The empirical evidence of color chaos is further supported by consistent results from complementary nonlinear tests of filtered HP cycles (Table III).

Table III

Table III. Characteristic statistics for stock market indicators

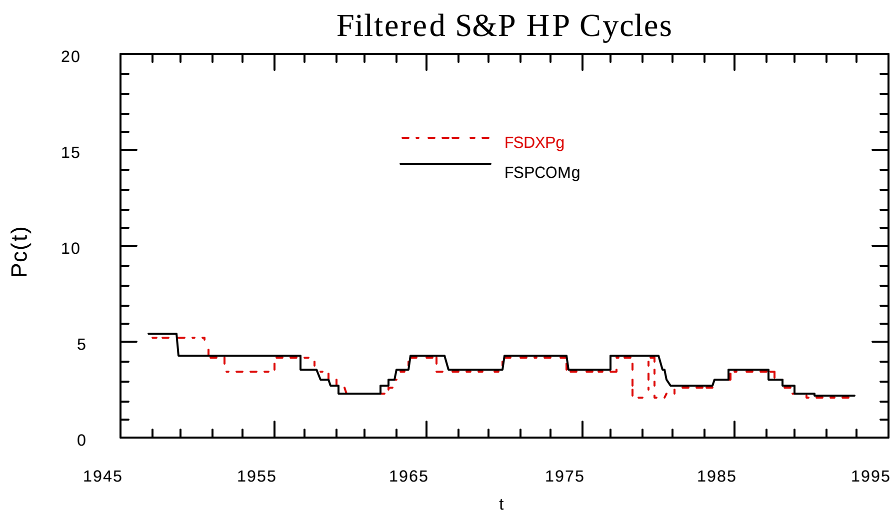
Data	η	υ(%)	CCgo	Pc	ϕ(%)	P_{dc}	λ^{-1}	μ
FSPCOM	0.828	68.6	0.847	3.6	25.9	3.3	5.0	2.5
FSDXP	0.804	64.6	0.829	3.5	27.7	2.9	6.9	2.4

Here, η is the ratio of standard deviations of the reconstructed series $S_g(t)$ over the original HP cycles $S_o(t)$; υ, their percentage ratio of variance; CC_{go}, their correlation coefficient [also in the Table II]. P_c, is the mean characteristic period from time-frequency analysis; P_{dc}, the decorrelation period from correlation analysis; ϕ, is the frequency variability (in time) measured by the percentage ratio of the standard deviation of f_c to the mean value of f_c over time evolution; λ is the Liapunov exponent, its reverse λ^{-1} is also a measure of a time scale, which is in the same range of P_{dc} for deterministic cycles. μ is the correlation dimension for attractors. All the time unit here is in years.

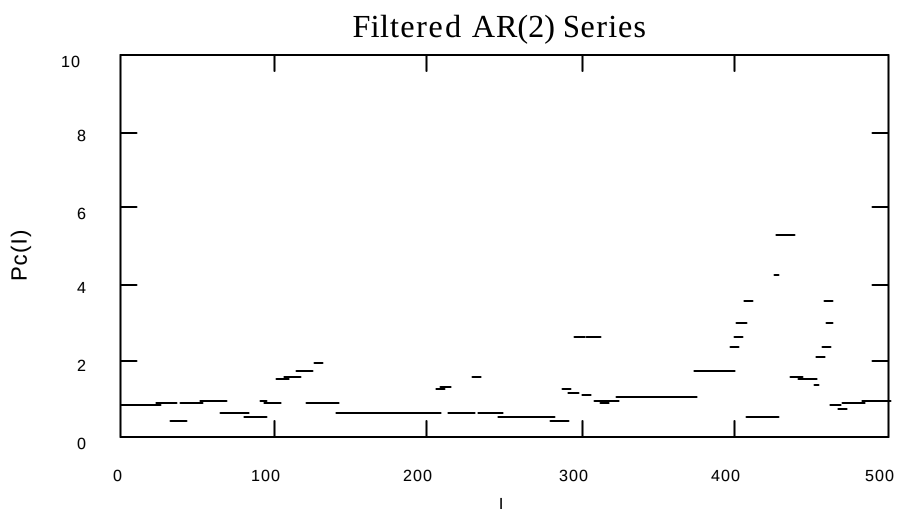
Characteristic frequencies of deterministic cycles are found in HP detrended cycles. Their frequency variability, measured by the ratio of standard deviation to mean frequency, are about 25 percent over a history of 45 years. The frequency stability of business cycles in the stock market is quite remarkable. The bandwidth of the characteristic frequency fc for HP cycles is just a few percent of the frequency span of white noise. This is strong evidence of economic color chaos even in a noisy and changing environment.

From Table III, we can see that FSPCOM and FSDXP are quite similar in frequency pattern and dimensionality. The characteristic period P_c from the time-frequency analysis and the decorrelation period P_{dc} from the correlation analysis are remarkably close. It is known that a long correlation is an indicator of deterministic cycles [Chen 1988, 1993]. However, time-frequency analysis provides a better picture of persistent cycles in business movements than correlation analysis and nonlinear analysis based on time-invariant representations.

The frequency patterns of the stock-market indexes disclose a rich history of market movements [Fig.6].



(6a). Frequency stability under historical shocks. The time path of characteristic period P_c for FSPCOM & FSDXP HP cycles. N=552.



(6b). Filtered AR(2) series. N=512. H=0.5.

Fig. 6. Time paths of instantaneous frequencies. Persistent cycles of FSPCOMln and FSDXP HP cycles have stable characteristic frequency over time. Filtered stochastic series has no trace of persistent cycles.

The extraordinary resilience of the stock market can be revealed from the stable frequency pattern under the oil-price shocks in 1973, 1979 and the stock-market crash in October 1987. These events generated only minor changes in the characteristic period P_c for FSPCOM and FSDXP indexes.

Economic historians may use the P_c path as a useful tool in economic diagnosis. After a close examination of Fig. 6, we found that the frequency shifts of S&P indexes occurred after the oil-price shock in 1973, but happened before the stock market crash in 1987. If we believe that the cause of an event always comes before the effect, then our diagnosis of these two crises would be different. The oil price shocks were external forces to the stock market, while the stock market crash resulted from an internal instability.

Our findings of nonlinear trend and persistent cycles reveal a rich structure from stock market movements. For example, the equity premium puzzle will have a different perspective because the frequency pattern of consumption and investment are not similar to that of stock-market indicators [Mehra and Prescott 1985, Chen 1996]. We will discuss the issue elsewhere.

VII. Risk, Uncertainty, and Information Ambiguity

Franck Knight made a clear distinction between risk and uncertainty in the market [Knight 1921]. Keynes also emphasized the unpredictable nature of "animal spirits" [Keynes 1921, 1936]. From the view of nonequilibrium thermodynamics, uncertainty is caused mainly by time evolution in open systems [Prigogine 1980].

The random-walk model of asset pricing has two extreme features. On one hand, the future price path is completely unpredictable. On the other hand, the average statistics are completely certain because the probability distribution is known and unchanged. According to equilibrium theory, only measurable risk with known probability exists in the stock market, no uncertainty with unknown and changing probability is considered in asset pricing models. The static picture of CAPM ignores the issue of uncertainty raised by Knight and Keynes.

Both practitioners and theoreticians are aware of the impact of business cycles. Fischer Black, the originator of the geometric random walk model in option pricing theory, made the following observations (the underline is added by the author) [Black 1990]:

> "One of the (Black-Scholes) formula's simple assumption is that the stock's future volatility is known and constant. Even when jumps are unlikely this assumption is too simple. <u>Perhaps the most striking thing I found was that volatilities go up as a stock prices fall and go down as stock price rise</u>. Sometimes a 10% fall in price means more than a 10% rise in volatility . . . . . . After a fall in the stock price, <u>I will increase my estimated volatility even where there is no increase in historical volatility</u>."

From Black's observation, the implied volatility, the only unknown parameter in option pricing theory, does not behave as a slow changing variable, that is a necessary condition for meaningful statistic concepts of mean and variance, but acts like a fast changing variable, such as trend-shifting and phase-switching in business cycles [see also Fleming, Ostdiek, and Whaley 1994]. Clearly, the up-trend or down-trend of price levels strongly influence the market behavior, even when historic variance may not change significantly. Black's observation of changing implied volatility helps our studies of nonlinear trends and business cycles in the stock market. We will discuss the issue in the near future.

In the equilibrium theory of the capital asset pricing model (CAPM), risk is represented by the variance of a known distribution of white noise. From our analysis, the risk caused by high frequency noise only accounts for about 30 percent of variance from FSPCOM and FSDXP HP cycles.

According to our analysis, there is an additional risk generated by a chaotic stock market. About 70 percent of variance from HP detrended cycles is associated with color chaos whose characteristic frequency is relatively stable. For the last 45 years, the variability of the characteristic period for FSPCOM and FSDXP is less than 30 percent. From this regard, the discovery of color chaos in the stock market indicates a limited predictability of turning points. We can develop a new program of period-trading rather than a level-trading strategy in investment decision and risk management. The frequency variability implies a forecasting error in a range of a fraction of the observed characteristic period. Clearly, the knowledge of HP cycles does little help for short-term speculators. Further study of higher-frequency data is needed for investors.

Recent literature of nonstationary time-series analysis such as ARCH and GARCH models focus on the issue of a changing mean and variance in the random-walk model with a drift. We found two more sources of uncertainty: changing frequency and shifting trend in an evolving economy. These uncertainties severely restrict our predictability of a future price trends and future frequency of business cycles. Therefore, we have a new understanding of the difficulties in economic forecasting.

In the two-dimensional landscape of time-frequency representation, there is no absolute dividing line between stochastic noise and deterministic cycles. The concept of perfect information and incomplete information can only be applied when the risk can be measured by a known distribution, such as a normal distribution in CAPM or a log-normal distribution in option pricing theory. The question of information ambiguity arises in signal processing when information is a mixture of deterministic and stochastic signals. Under the Wigner distribution, excess information with an infinite order of D coupling produces misleading interferences and false images. The real challenge in pattern recognition is searching relevant information from conflicting news and experiences. For example, the merger and acquisition in the capital market is a war game in the business world filled with conflicting and false information. That is why the stock market often overreacts to market news on merger and acquisition.

From our analysis of historical events, the time path of stock prices is not a pure random-walk. Price history is a rich source of new information if we have the right tools of signal decoding. In balance, our approach of trend-cycle decomposition and time-frequency analysis increases a limited predictability of chaotic business cycles, and at the same time reveals two more uncertainties in nonlinear trends and evolving frequency.

The equilibrium school in finance theory emphasizes the forecasting difficulty caused by noisy environments, but ignores the uncertainty problem in evolving economies.

VIII. Persistent Cycles and The Friedman Paradox

A strong argument against the relevance of economic chaos comes from the belief that economic equilibrium is characterized by damped oscillations and absence of deterministic patterns. Friedman asserted that market competition will eliminate the destabilizing speculator, and speculators will lose money [Friedman 1953]. Friedman did not realize that arbitrage against a market sentiment is very risky if rational arbitrageurs have only limited resources [Shleifer and Summers 1990]. Friedman also assumed that winner-followers could perfectly duplicate winner's strategy. This could not be done for chaotic dynamics in an evolving economy.

People may ask what will happen once the market knows about the limited predictability of color chaos in the stock market? At this stage, we can only speculate the outcome under complex dynamics and market uncertainty. We believe that the profit opportunities associated with color chaos are limited and temporary, but the nonlinear pattern of persistent cycles will remain in existence and perhaps evolve over time.

Based on our previous discussion, we will point out two likely outcomes: co-existence of diversified strategies and persistence of chaotic cycles. There is no way to have a sure winner, because of trend uncertainty and information ambiguity. Nonlinear overshooting and time-delay in feedback may actually create the chaotic cycles in the market dynamics [Chen 1988, 1993, Wen 1993].

There are several factors that may prevent wiping out the persistent pattern of color chaos. First, people are incapable in distinguishing fundamental movements and sentimental movements in price changes, especially when facing a growing trend. The same argument on a monetary veil of real income caused by inflation can be applied to a price veil of stock value caused by a changing market sentiment along with an evolving economic growth. Second, information ambiguity is caused by a limited time-horizon in observation of complex systems. Bounded rationality is rooted not only in limited computational capacity, but also in dynamic complexity [Prigogine 1993].

Winner-following or trend-chasing behavior may change the amplitude or frequency of a color chaos, but the chaotic pattern will persist in a nonlinear and non-equilibrium world.

IX. Conclusions

There is no question that external noise and measurement errors always exist in economic data. The questions are whether some deterministic pattern and dynamical regularities are observable from the economic indicators and whether economic chaos is relevant in economic theory [Granger and Teräsvirta 1993]. Our answer is yes if the color-chaos model is addressing the empirical pattern of business cycles.

From our empirical analysis, stock market movements are not pure random-walks. A large part of stock-price variance can be explained by a color-chaos model of business cycles. Its characteristic frequency is in the range of business cycles. The frequency stability of the stock market is remarkable under historical shocks. The existence of persistent chaotic cycles reveals a new perspective of market resilience and new sources of economic uncertainties. To observe chaotic patterns of business cycles, a proper trend-cycle decomposition and a proper time-window is the key in economic signal processing. We need a modified theory of asset pricing in a chaotic stock market.

A new way of thinking needs new representation. From business practice, it is known that the time-window plays a critical role in evaluating key statistics, such as mean, variance, and correlations in asset pricing. Under a coherent wave representation, such as the case in quantum mechanics and information theory, the frequency-window is closely related to the time-window according to the uncertainty principle [Gabor 1946]. That is why the joint time-frequency representation is essential for time-dependent signal processing.

Like a telescope in astronomy or a microscope in biology, time-frequency analysis opens a new window of observing evolving economies. As a building block of nonlinear economic dynamics, the color-chaos model of stock-market movements may establish a link between business-cycle theory and asset-pricing theory.

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