Empirical and theoretical evidence of economic chaos

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Abstract

Empirical and theoretical investigations of chaotic phenomena in macroeconomic systems are presented. Basic issues and techniques in testing economic aggregate movements are discussed. Evidence of low dimensional strange attractors is found in several empirical monetary aggregates. A continuous time deterministic model with delayed feedback is proposed to describe the monetary growth. Phase transition from periodic to chaotic motion occurs in the model. The model offers an explanation of the multiperiodicity and irregularity in business cycles and of the low-dimensionality of chaotic monetary attractors. Implications in monetary control policy and a new approach to forecasting business cycles are suggested.

In recent years, there has been rapid progress in the studies of deterministic chaos, random behavior generated by deterministic systems with low dimensionality. This progress has been made not only in theoretical modelling, but also in experimental testing [Abraham , Gollub, and Swinney 1984]. Chaotic models have been applied to a variety of dynamic phenomena in the areas of fluid dynamics, optics, chemistry, climate and neurobiology. Applications to economic theory have also been developed, especially in business cycle theory [see review article: Grandmont and Malgrange 1986].

Over the last century, the nature of business cycles has been one of the most important issues in economic theory [Zarnowitz 1985]. Business cycles have several puzzling features. They have elements of a continuing wave-like movement; they are partially erratic and at the same time serially correlated. More than one periodicity has been identified in business cycles in addition to long growth trends. Most simplified models in macroeconomics address one of these features [Rau 1974] , while system dynamics models describe economic movements in terms of a large number of variables [Forrester 1977].

Two basic questions arise in studies of business cycles. Are endogenous mechanism or exogenous stochastics the main cause of economic fluctuations? And can complex phenomena be characterized by mathematical models as simple as, say, those for planetary motion and electricity?

The early deterministic approach to business cycles with well-defined periodicity mainly discussed the endogenous mechanism of economic movements. A linear deterministic model was first proposed by Samuelson [1939], which generated damped or explosive cycles. Nonlinearities were introduced in terms of limit cycles to explain the self-sustained wavelike movement in economics [Goodwin 1951].

A stochastic approach seems to be convenient for describing the fluctuating behavior in economic systems [Osborne 1959; Lucas 1981]. The problem with the stochastic models, however, lies in the fact that random noise with finite delay terms (usually less than ten lags, in practice) only explains the short term fluctuating behavior. Most aggregate economic data are serially correlated not only in the short term but also over long periods. Two methods dealing with long correlations are often used: longer lags in regression studies and multiple differencing time series in ARIMA models. Longer lags require estimating more "free parameters", while ARIMA models are essentially whitening processes that wipe out useful information about deterministic mechanism.

Actually, fluctuations may be caused by both intrinsic mechanism and external shocks. An alternative to the stochastic approach with a large number of variables and parameters, is deterministic chaos, with few variables or low-dimensional strange attractors [Schuster 1984]. This is the approach adopted in the present article. Newly developed numerical techniques of nonlinear dynamics also shed light on a reasonable choice of the number of variables needed in characterizing a complex system.

An increasing number of works examine economic chaos. Most theoretical models are based on discrete time [Benhabib 1980; Stutzer 1980; Day 1982; Grandmont 1985; Deneckere and Pelikan 1986; Samuelson 1986], only one long wave model is based on continuous time [Rasmussen et al 1985]. On-going empirical studies are conducted by a few economists [Sayers 1986; Brock 1986; Scheinkman and Le Baron 1987; Ramsey and Yuan 1987; Frank and Stengos 1987]. Some clues of nonlinearities have been reported, but no solid evidence of chaos has yet been found by these authors. Two efforts were made to fit nonlinear discrete models with empirical data [Dana and Malgrange 1984; Candela and Gardini 1986], but the parameters were found outside the chaotic regions.

We started search for empirical evidence of chaos in economic time series in 1984. The main features of deterministic chaos, such as complex patterns of phase portraits and positive Lyapunov exponents, have been found in many economic aggregate data such as GNP and IPP, but most of our studies have failed to identify the dimensionality of attractors because of limited data. Then we tested monetary aggregates at the suggestion of W. A. Barnett. Low-dimensional strange attractors from weekly data were found in 1985, and a theoretical model of low-dimensional monetary attractors was developed in 1986 [Chen 1987]. A brief description of comprehensive studies of economic chaos is presented here for general readers.

In this article, a short comparison between stochastic and deterministic models is introduced. Positive evidence of low-dimensional strange attractors found in monetary aggregates is shown by a variety of techniques. A continuous-time model is suggested to describe the delayed feedback system in monetary growth. The period-doubling route to chaos occurs in the model [Feigenbaum 1978]. The model offers an explanation for the low dimensionality of chaotic monetary time series and for the nature of business cycles and long waves. Finally, the implications of deterministic chaos in economics and econometrics are discussed.

Simple pictures of deterministic and stochastic processes

To what extent economic fluctuations around trends should be attributed to endogenous mechanism (described by deterministic chaos) or exogenous shocks (described by stochastic noise) is a question that can be addressed by empirical tests.

There are at least four possible candidates in describing fluctuating time series: linear stochastic process, discrete deterministic chaos, continuous deterministic chaos, and nonlinear deterministic chaos plus noise. The test of the last one is only in its infancy, because a high level of noise will easily destroy the subtle signal of deterministic chaos. We mainly discuss the first three candidates here and give numerical examples of white noise and deterministic chaos as the background for further discussions. The linear autoregressive AR(2) model adopted in explaining the fluctuations of log linear detrended GNP time series [Brock 1986] is demonstrated as an example of a linear stochastic process. For deterministic chaos, two models are chosen: the discrete logistic model [May 1976], which is widely used in population studies and economics, and the continuous spiral chaos model [Rossler 1976].

The time sequences of these models are shown in Figure 1. They seem to be equally capable to describe economic fluctuations when appropriate scales are used to match real time series. But closer examination reveals the differences among them.

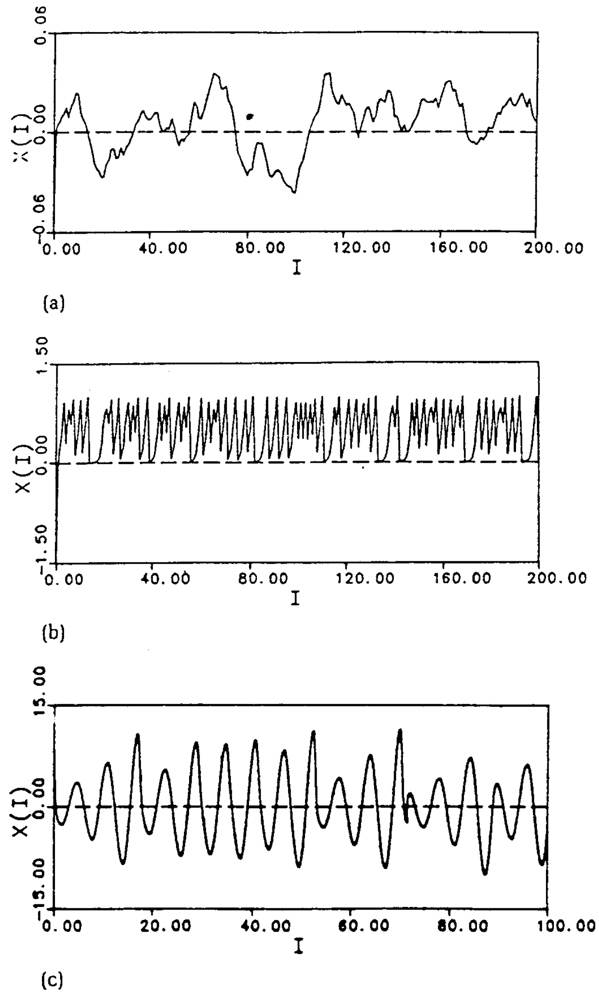


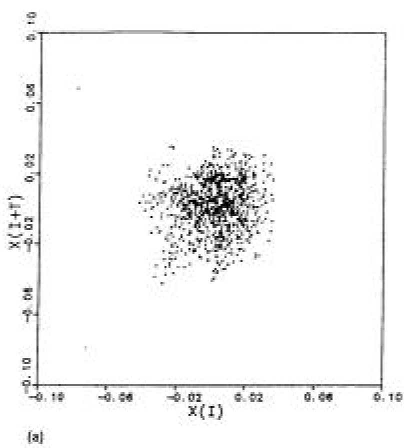
Fig. 1. Comparison of the time series of model solutions. The time units are arbitrary. (a) AR(2) linear stochastic model. (b) Discrete logistic chaos generated by mapping $X(t + 1) = 4X(t)[1 - X(t)]$. (c) Rössler model of spiral chaos with time interval $dt = 0.05$.

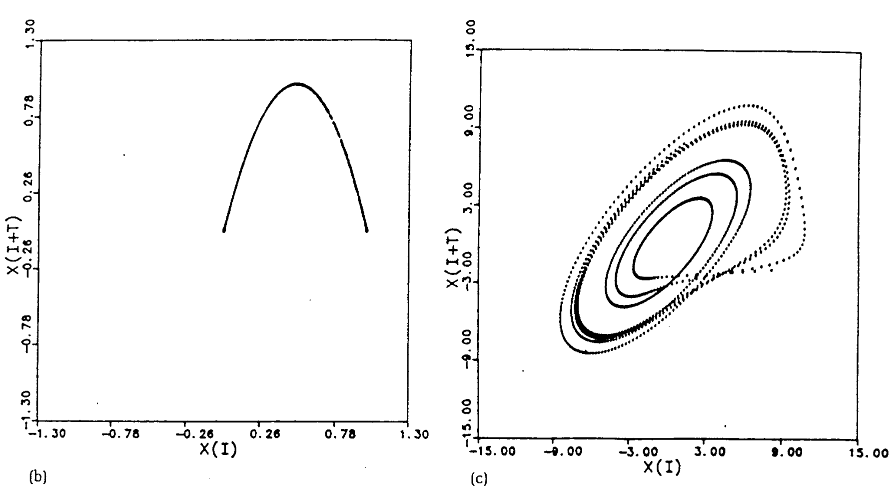
Phase space and phase portrait

From a given time series X(t), an m-dimensional vector V(m,T) in phase space can be constructed by the m-history with time delay T: V(m,T) = {X(t), X(t+T), . . , X[t+(m-1)T] }, where m is the embedding dimension of phase space [Takens 1981]. This is a powerful tool in developing numerical algorithms of nonlinear dynamics, since it is much easier to observe only one variable to analyze a complex system.

The phase portrait in two-dimensional phase space X(t+T) versus X(t) gives clear picture of the underlying dynamics of a time series. With the fixed point solution (the so-called zero-dimensional attractor), the dynamical system is represented by only one point in the phase portrait. For periodic solution (the one-dimensional attractor), its portrait is a closed loop. Figure 2 displays the phase portrait of the three models. The nearly uniform cloud of points in Figure 2a closely resembles the phase portrait of random noise (with infinite degree of freedom). The curved image in Figure 2b is characteristic of the one-dimensional unimodal discrete chaos. The spiral pattern in Figure 2c is typical of a strange attractor whose dimensionality is not an integer. Its wandering orbit differs from periodic cycles.

Fig. 2. Comparison of the phase portraits of model solutions. N = 1,000. (a) AR(2) model with T = 20. (b) Logistic chaos with T = 1. (c) Rössler model with T = 1 and dt = 0.05.





Long-term autocorrelations

The autocorrelation function is another useful concept in analyzing time series. The autocorrelation function AC(I) is defined by

$$AC(I) = AC(t'-t) = \mathrm{cov}\,[\,X(t'), X(t)\,] \,/\, E\,[\,(X(t) - M)^2] \qquad (1)$$

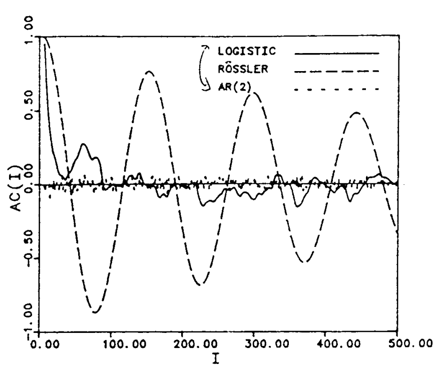
Where M is the mean of X(t) and cov[X(t'), X(t)] is the covariance between X(t') and X(t). They are given by

$$M = E[\,X(t)\,] = \{\,\sum_{t=1}^{N} X(t)\,\} \,/\, N \qquad (2)$$

$$\mathrm{cov}\,(\,X(t'), X(t)\,) = E\,[\,(X(t') - M)\,(\,X(t) - M)\,] \qquad (3)$$

It is known that the autocorrelation function of the periodic motion is periodic and that of the white noise is a delta function. Figure 3 shows autocorrelations of the AR(2) process quickly decay to small disturbances. The autocorrelations of the logistic chaos look the same as those of white noise. The Rossler attractor displays some resemblance to periodic cycles. Its autocorrelations have initial exponential decay after a characteristic decorrelation time $T_{d,}$ followed by wave-like fat tails. T_d is determined by the first vanishing autocorrelations.

Fig. 3. Comparison of the autocorrelations of the three model solutions with 1,000 data points. The time units are the same as in Figure 1.



Testing economic chaos in monetary aggregates

Testing economic aggregate time series is a complex process, since they contain growth trends. Not all the techniques in nonlinear dynamics developed by mathematicians and physicists [Mayer-Kress 1986] are applicable to economic time series. For example, Poincare sections and spectral analysis used in physics require more than several thousand data points. Data quantity and data quality are crucial in applying the currently existing techniques.

After introducing the monetary indexes, we focus on the testing and modelling of monetary aggregates. The data source is Fayyad [Fayyad 1986].

Monetary aggregates

Observable indicators are essential to empirical investigations. In simple physical systems, some macroscopic quantities (such as mass and energy) can be simple summation of microscopic quantities. The right choice of aggregate indexes for economic system remains an issue on which there is no consensus. For example, there are 27 component monetary assets - currency, traveler checks, demand deposit, Eurodollars, money market deposit, saving deposit, Treasury securities, commercial paper, and so on - according to the Federal Reserve's latest classification of the American monetary system. Four levels of simple-sum aggregate indexes, M1, M2, M3 and L, consisting of 6 to 27 monetary assets, are used by the Federal Reserve. There are also parallel theoretical indexes, such as Divisia monetary aggregates, initiated by W. A. Barnett. Better aggregate indexes are needed to describe macroeconomic movements by simple mathematical methods.

We tested 12 types of monetary index time series including official simple-sum monetary aggregates (denoted by SSM), Divisia monetary demand aggregates (DDM), and Divisia monetary supply aggregates (DSM); each yielded about 800 weekly data points between 1969 and 1984. Five of them were successful in testing strangeness: simple-sum SSM2, Divisia demand DDM2, DDM3, DDL and Divisia supply DSM2 monetary aggregates. The behaviors of Divisia aggregates are very similar. We only discuss SSM2 and DDM2 here for brevity. The exponential growth trends of these time series are shown in Figure 4.

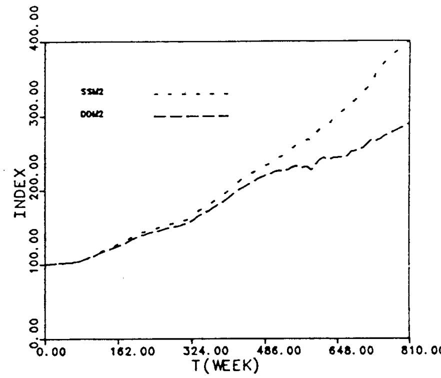


Fig. 4. The exponential growth trends in time series of monetary aggregates: official simple-sum index SSM2 and Divisia index DDM2 (January 1969–July 1984). The time unit is one week.

Observation reference and first-difference detrending

Mathematical models with attractor solutions can greatly simplify descriptions of complex movements without obvious growth trends. The choice of detrending methods basically is a choice of reference system or transformation theory. Detrending is a solved problem for physicists when observations of physical systems are conducted in appropriate inertial reference systems. However, it is an unsolved issue in testing economic time series. How to choose a reference system to observe the global features of economic movements is a critical question for identifying the deterministic mechanisms of economic activities. We attempt to answer this question through numerical experiments on empirical data.

The percentage rate of change and its equivalent form, the logarithmic first differences, are widely used in fitting stochastic econometric models [Osborne 1959; Friedman 1969]. It can be defined as follows:

$$Z(t) = \ln S(t+1) - \ln S(t) = \ln \{ S(t+1) / S(t) \} \qquad (4)$$

where S(t) is the original time series, and Z(t) is the logarithmatic first difference. Its ineffectiveness for observing chaos will be shown later.

Log linear detrending and growth cycles

We detrended data using log linear detrending which was suggested by W. A. Barnett. The same detrending was also been used by other economists [Dana & Malgrange 1984; Brock 1986]. In log linear detrending, we have

$$X(t) = \ln S(t) - (k_0 + k_1 t) \quad (5)$$

or

$$S(t) = S_0 \exp (k_1 t) \exp (X(t)) \quad (6)$$

where S(t) is the original time series, and X(t) is the resulting log linear detrended time series, k_0 is the intersection, k_1 the constant growth rate, and $S_0 = \exp(k_0)$.

After numerical experiments on a variety of detrending methods and economic time series, we finally found that the percentage rate of change and its equivalent methods are whitening processes based on short time scaling. Log linear detrending, on the other hand, retains the long term correlations in economic fluctuations, since its time scale represents the whole period of the available time series. Findings of evidence of deterministic chaos mainly from log linear detrended economic aggregates lead to this conclusion. Figure 5a shows the time sequences of the log linear detrended (denoted by LD) monetary aggregates SSM2. Its almost symmetric pattern of nearly equal length of expansion and contraction is a typical feature of growth cycles in economic systems. The usual business cycles are not symmetrical, their longer expansions and shorter contractions can be obtained by superimposing a trend with constant growth rate adding to the symmetric growth cycles. The logarithmic first - difference time series (denoted by FD) SSM2 is given in Figure 5b as a comparison. The latter is asymmetric and more erratic.

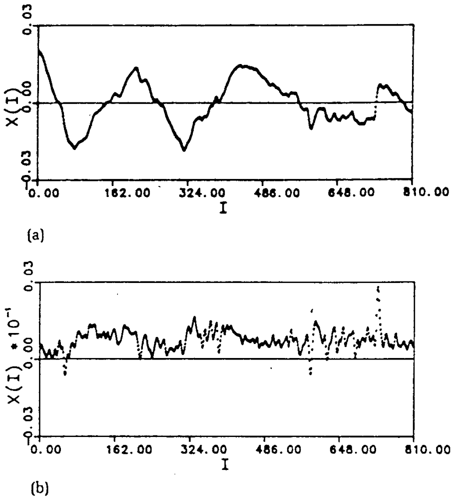


Fig. 5. Comparison of the detrended weekly time series SSM2. (a) Symmetric LD SSM2: the log linear detrended SSM2 with a natural growth rate of 4 percent per year. (b) Asymmetric FD SSM2: the logarithmic first differences of SSM2.

Empirical evidence of deterministic and stochastic processes

Based on the phase portrait and autocorrelation analysis, we can easily distinguish qualitatively a stochastic process from a deterministic one. A comparison between IBM daily stock returns and monetary aggregates follows.

Figure 6a presents the phase portrait of detrended monetary aggregates LD SSM2. It rotates clockwise like the spiral chaos in Fig. 2c. The complex pattern is a potential indication of nonlinear deterministic movements and eliminates the possibilities of white noise or simple periodic motions. The phase portrait of IBM daily stock returns is shown in Figure 6b. It closely resembles Gaussian white noise. It is consistent with previous findings in economics [Osborne 1959; Fama 1970]. The autocorrelations of the detrended time series are shown in Figure 7. Readers may compare these with the autocorrelations in Figure 3.

If we approximate the fundamental period T_1 by four times the decorrelation time T_d, as in the case of periodic motion, then, T_1 is about 4.7 years for LD SSM2, which is very close to the common experience of business cycles. We will return to this point later.

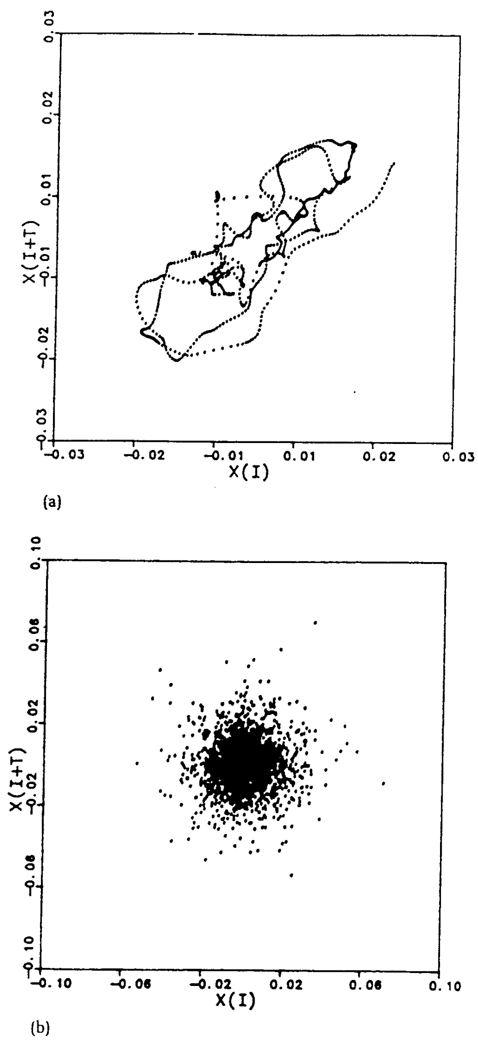


Fig. 6. Comparison of the phase portraits of empirical time series. Time delay T = 20. (a) LD SSM2 time series. The time unit is one week. N = 807 points. (b) IBM daily common stock returns. The time interval is one day. N = 1,000 points, beginning on July 2, 1962.

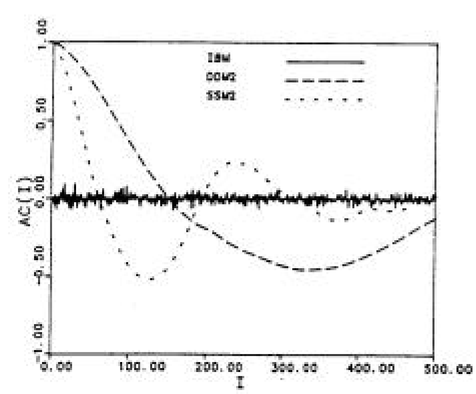


Fig. 7. Comparison of autocorrelation functions: AC(I) plotted against I. There are three time series: LD SSM2, LD DDM2, and IBM daily stock returns, each in their original time units. N = 807.

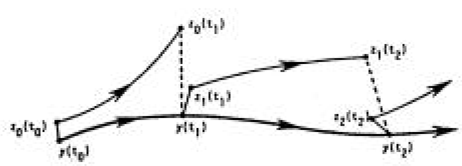
The numerical maximum Lyapunov exponent

Chaotic motion is sensitive to initial conditions. Its measure is the Lyapunov exponents, which are the average exponential rates of divergence or convergence of nearby orbits in phase space. Consider a very small ball with radius $\in(0)$ at time $t = 0$ in the phase space. The ball may distort into an ellipsoid as the dynamical system evolves. Let the length of the i-th principal axis of this ellipsoid at time t be $\in_i(t)$. The spectrum of Lyapunov exponents λ_i from an initial point can be obtained theoretically by [Farmer 1982]

$$\lambda_i = \lim_{t \to} \lim_{\in(0) \to 0} \{ \ln [\in_i(t) / \in_i(0)] / t \} \qquad (7)$$

The maximum Lyapunov exponent λ (the largest among λ_i) can be calculated numerically by the Wolf algorithm [Wolf et al. 1985] where the limiting procedure is approximated by an averaging process over the evolution time EVOLV. This algorithm is applicable when the noise level is small. A sketch of the algorithm is shown in Figure 8. The maximum Lyapunov exponent λ is negative for stable systems with fixed points, zero for periodic or quasiperiodic motion, and positive for chaos.

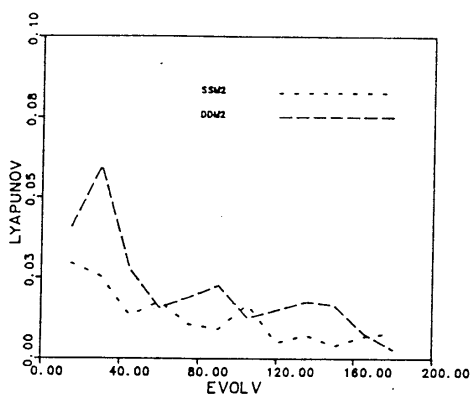
Fig. 8. An artist's sketch of the Wolf algorithm. The lower line, y(t), is the reference orbit. The upper broken line, z(t), starting in the neighborhood of $y(t_0)$, is traced to calculate the divergence of the lines. The points $z_0(t_1)$, $z_1(t_2)$, . . . are replaced by new nearest neighboring points $z_1(t_1)$, $z_2(t_2)$, . . . after an evolution time EVOLV for numerical calculation.



In theory, the maximum Lyapunov exponent is independent of the choice of evolution time EVOLV, embedding dimension m and time delay T. In practice, the value of Lyapunov exponent does relate to the numerical parameters. The range of evolution time

EVOLV must be chosen by numerical experiments. The positive maximum Lyapunov exponents of the investigated monetary aggregates are stable over some region in evolution time shown in Figure 9. The numerical Lyapunov exponent is less sensitive to the choice of embedding dimension m. In our tests, we fixed m at 5 and time delay T at 5 weeks based on the numerical experiments. For example, the stable region of EVOLV is 45-105 weeks for SSM2 and 45-150 weeks for DDM2. Their average maximum Lyapunov exponent λ over this region are 0.0135 and 0.0184 (bit per week), respectively.

Fig. 9. The maximum Lyapunov exponents of log linear detrended monetary aggregates LD SSM2 and LD DDM2. The maximum Lyapunov exponents of monetary aggregates plotted against the evolution time EVOLV, calculated in phase space with time delay T = 5 weeks and embedding dimension m = 5. The unit is bit per week. The evolution time EVOLV in calculating numerical Lyapunov exponents varies from 15 to 180 weeks at 15-week intervals.



The characteristic decorrelation time T_d of the LD SSM2 is 61 weeks. The reciprocal of the maximum Lyapunov exponent λ^{-1} (= 74.1) for LD SSM2 is roughly of the same order of magnitude as the decorrelation time T_d [Nicolis and Nicolis 1986]. This relation does not hold for pure white noise.

The correlation dimension

The most important characteristic of chaos is its fractal dimension [Mandelbrot 1977] which provides a lower bound to the degrees of freedom for the system [Grassberger and Procaccia 1983, 1984]. The popular Grassberger-Procaccia algorithm estimates the fractal dimension by means of the correlation dimension D. The correlation integral $C_m(R)$ is the number of pairs of points in m-dimensional phase space whose distances between each other are less than R. For random or chaotic motion, the correlation integral $C_m(R)$ may distribute uniformly in some region of the phase space and has a scaling relation of R^D. Therefore, we have

$$\ln_2 C_m(R) = D \ln_2 R + \text{constant} \quad (8)$$

For white noise, D is an integer equal to the embedding dimension m. For deterministic chaos, D is less than or equal to the fractal dimension. The Grassberger-Procaccia plots of $\ln C_m(R)$ versus $\ln R$ and slope versus log R for LD SSM2 and LD DDM2 are shown in Figures 10 and 11. For R too large, $C_m(R)$ becomes too saturate at the total number of data points (see the right-hand regions of Figures 10 and 11). For R too small, the algorithm detects the noise level of the data (see the left-hand regions of Figures 10 and 11). The existence of linear regions of intermediate R, which reflect the fractal structure of the attractors, is shown in Figures 10a and 11a. The correlation dimension can be determined from the the saturated slope of the plateau region in Figures 10b and 11b.

Fig. 10. The Grassberger-Procaccia plots for calculating the correlation dimension of LD SSM2 time series with time delay T = 8. The embedding dimension m = 2, . . . , 6 is taken as a parameter. (a) Plots of $\ln_2 C_m(R)$ versus $\ln_2 R$. The plots rotate downwards and to the right as m increases. (b) Plot of the slopes of the curves in (a) against $\ln_2 R$. The linear region of the curves in (a) can be identified from the plateau region in (b). The correlation dimension is equal to the saturated slope 1.8 for LD SSM2 measured from the plateau region with m = 3.

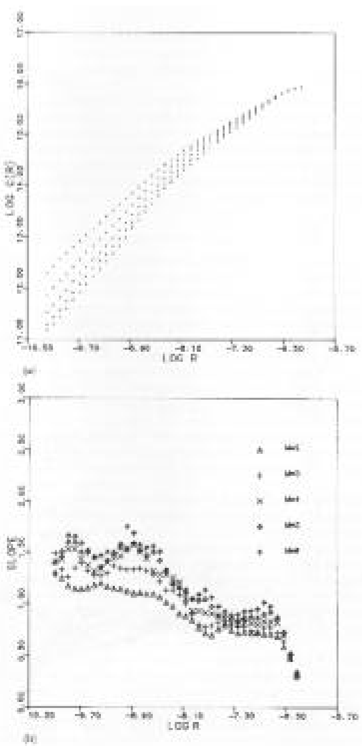
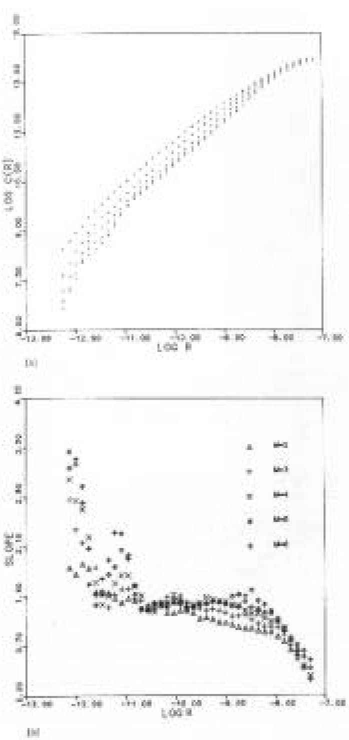


Fig. 11. (a) and (b) The Grassberger-Procaccia plots for calculating the correlation dimension of LD DDM2. The correlation dimension is 1.3.



We found that the correlation dimensions of the investigated five monetary aggregates, including four Divisia monetary indexes and one official simple-sum monetary index, were between 1.3 and 1.5. For other monetary aggregates, no correlation dimension could be determined. These findings are consistent with previous studies in economic aggregation theory and index number theory, which indicate that, except for SSM2, Divisia monetary aggregates are better indexes than simple-sum monetary aggregates [Barnett, Hinich, and Weber 1986; Barnett and Chen, 1988].

Some remarks about numerical algorithms

Given a deterministic attractor whose correlation dimension is D, we first ask how many data points are needed to determine the dimensionality D [Greenside et al. 1982]. The minimum data points N_D with a D-dimensional attractor can be estimated by scaling relation h^D, where the constant h varies with attractors. Practically, we can only identify low-dimensional attractors with finite data sets, since N_D increases exponentially with D. For the Mackey-Glass model [1977], 500 points are needed for D=2 and more than 10,000 points for D=3. In the Couette-Taylor experiment, N_D is about 800 points for D=2.4, 40,000 points for D=3, and 50 billion points for D=7 [Brandstater and Swinney 1987]. This issue seems to be ignored by some economists. For example, in Brock [1986], the correlation dimension of GNP with 143 quarterly data points was calculated under the extremely high embedding dimension (m=20) without showing the linear region of the Grassberger-Procaccia plots. In our experience, the width of the linear region shrinks rapidly to zero when m increases beyond 6, as seen in Figure 10 and 11. Practically, m is large enough when m reaches 2D+1.

There is another concern about the time expansion covered by the time series. In physics experiments, the sampling rate is typically 10-100 points per orbit. Therefore, 100-1000 periods are needed for D=3 and 5-50 periods for D=2. We tested this estimation in terms of the Mackey-Glass attractor. When the time delay †t is 17, its correlation dimension D is 1.95 calculated with 25,000 points [Grassberger and Procaccia 1983]. To compared this result, we estimated the correlation dimension under a variety of sampling rates and time periods. We find the error is within 1 percent with

100 periods, 3 percent with 30 periods, 8 percent with 10 periods, and 18 percent with 5 periods when using 1000-3000 data points. Similar results are obtained for the model we develop later.

It should be noted that there is no unique approach to identify deterministic chaos with certainty. Several algorithms that may be complementary were used in our tests. At present, with only hundreds of data points, the discovery of economic strange attractors whose dimensionality is higher than 3 is unlikely.

We can only speculate why we were unable to identify correlation dimensions for other types of economic time series, such as GNP, IPP, and the Dow-Jones indexes, in our numerical tests. Either their dimensions are too high to be estimated for limited data, or their noise levels are too large to recover the subtle information of deterministic chaos.

A delayed feedback model of economic growth

Let us consider modelling the low-dimensional monetary strange attractors as growth cycles. There are several problems to be solved: time scale, dynamic mechanism, and system stability.

Continuous versus discrete time

Current economic studies are dominated by discrete models. Economists favor discrete models because economic data are often reported discretely in years, quarters or months, and because discrete models are easier for numerical regression. However, continuous-time models are needed when the serial correlation of disturbances can no longer be neglected [Koopmans 1950]. The decorrelation time T_d of the autocorrelations of time series sets a lower bound to the time unit of the discrete model. For a typical discrete model, T_d is in approximately the same length as the discrete time unit. The decorrelation time T_d for monetary attractors is more than 60 weeks. The time scales of discrete models of deterministic chaos with one or two variables in business cycle theory [Benhabib 1980; Day 1982; Grandmont 1985] are usually larger than the time scale of real business cycles [Sims 1986; Sargent 1987]. Clearly, the simple discrete model is not appropriate to describe monetary growth cycles. A continuous model is needed for the monetary time series.

The observed low correlation dimension of monetary aggregates sets additional constraints to the theoretical modelling of growth cycles. The minimum number of degrees of freedom required for chaotic behavior in autonomous differential equations is 3 [Ott 1981], so the fractal dimension will be larger than 2. Therefore, the driven oscillator in the long-wave model [Rasmussen, Mosekilde, and Sterman 1985] is not applicable in our case.

After comparing the correlation dimension and the phase portraits of existing models, we believe that the differential-delay equation is a good candidate for modelling monetary growth. For simplicity, we consider only one variable here. The low dimensionality of monetary attractors leads to the belief of the separability of the monetary deviations from other macroeconomic movements that are integrated in the natural trends of monetary growth rate.

Deviations from the trend and feedback behavior

The apparent monetary strange attractors are mainly found in log linear detrended data. This is an important finding to study control behavior in monetary policy. We believe that the human ability to manage information is limited even if decision makers have "perfect information". Economic behavior is more likely following some simple rule or procedure than providing global optima [Simon 1979]. We assume that the general trends of economic development, the natural growth rate, are perceived by people in economic activities as a common psychological reference or as the anchor in observing and reacting [Tversky and Kahneman 1974]. Administrative activities are basically reactions to deviations from the trend. We choose the deviation from the natural growth rate as the main variable in the dynamic model of monetary growth.

There are a number of differential-delay models in theoretical biology and population dynamics [Mackey and Glass 1977; May 1980; Blythe, Nishbet, and Gurney 1982; Chow and Green 1985]. Our model has a new feature which differs from previous models of population dynamics: its wave pattern should be symmetric, because we are dealing with detrended growth cycles. The wave form of business cycles is not symmetric, since they are observed in terms of the first difference of logarithmatic macroeconomic indexes or annual percent rate of growth. In an economic system moving with a constant growth rate, we define the reference equilibrium state as zero. The proposed equation is:

$$dX(t)/dt = a\, X(t) + F(\, X(t-\tau)\,) \qquad (9)$$

$$F(X) = X\, G(X) \qquad (10)$$

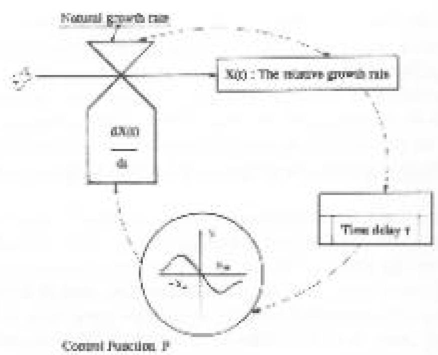
X is here the relative growth index, which measures the deviation from the trend. τ is the time delay, a is the expansion speed, F is the control function, and G is the feedback function.

There are two competing mechanisms in the growth system. The first is the stimulative growth that is an instantaneous response to market demand. It is described by the first term on the right of Eq. 9. A linear term for exponential growth is used for mathematical convenience. The second term represents the endogenous system control described by the control function F. This consists of feedback signal $X(t-\tau)$ and feedback function G. The time delay τ exists in the feedback loop because of information and regulation lags.

The flow diagram and the symmetric control function

Figure 12 shows a flow diagram to describe our model. There are several considerations in specifying F and G. We assume the control function F(X) has two extrema at $\pm X_m$ for the control target floor and ceiling [Solomon 1981]. G(X) should be nonlinear and symmetric, $G(-X) = G(X)$, in order to describe the overshooting in economic management and the symmetry in growth cycles. These features are essential to generate complex behavior in the economic growth model.

Fig. 12. The flow diagram for a delayed feedback system of economic growth.



In choosing the form of G, we do not use the polynomial function adopted in previous models with relaxation oscillations. Here, we suggest a simple exponential function to describe negative feedback reactions.

$$G(X) = - b \exp(- X^2 / \sigma^2) \qquad (11)$$

where b is the control parameter, σ is the scaling parameter, and the extrema of F(X) are located at $X_m = \pm \sigma /$ 2. Substituting Eqs. 10 and 11 into Eq.9 gives the following differential-delay equation:

$$dX(t)/dt = a X(t) - b X(t-\tau) \exp(-X(t-\tau)^2 / \sigma^2) \qquad (12)$$

We may change the scale by $X=X'\sigma$ and $t=t'\tau$, then drop the prime for convenience:

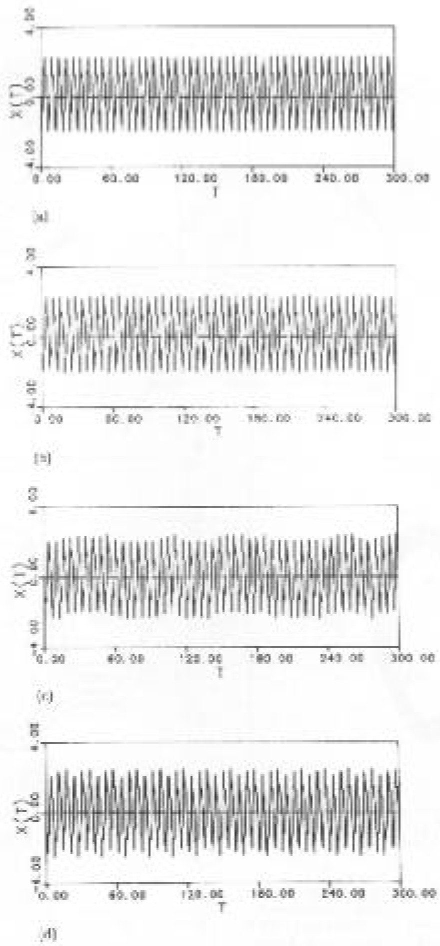
$$dX(t)/dt = a\tau X(t) - b\tau X(t-1) \exp(-X(t-1)^2) \qquad (13)$$

The rough behavior of the time delay Eq.13 can be discussed in terms of linear stability analysis in determining the boundaries of damped and divergent oscillations in the parameter space.

The period-doubling route to chaos

We solved Eq.13 numerically by the predictor-corrector approach. Time sequences and phase portraits of solutions with different b for fixed a and τ are shown in Figures 13 and 14. In order to identify the route to chaos, the power spectra are shown in Figure 15. The period-doubling route to chaos is observed when parameter changes induce bifurcations [Feigenbaum 1978]. One observes the fundamental frequency f_1 and its subharmonic frequency f_2 before and after transition to chaos in Figure 15c). In addition to period-1 orbit P1 (limit cycle) in Figure 14a, period-2 orbit P2 in Figure 14b and period-3 orbit P3 in Figure 14d, we also observe P4, P8 and P6 in the regions close to P2 and P3, respectively. The period-doubling route to chaos has also been found in other differential-delay models with asymmetric solutions [May 1980].

Fig. 13. The time sequences of the numerical solutions of Eq. 13. The parameters were fixed at $\sigma = 0.1$ and $\tau = 1$ while changing the parameter b. (a) Period-1 solution P1 (limit cycle) with $b = 5.7$. Its cycle number is C1. (b) Period-2 solution P2 (C2) with $b = 5.8$. (c) Chaotic solution CH with $b = 6.0$. (d) Period-3 solution P3 (C3) with $b = 6.3$.



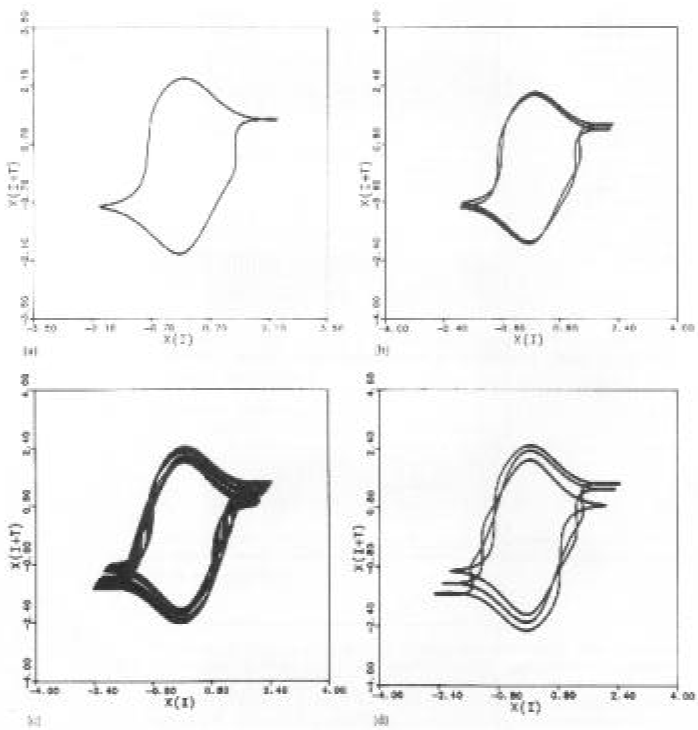
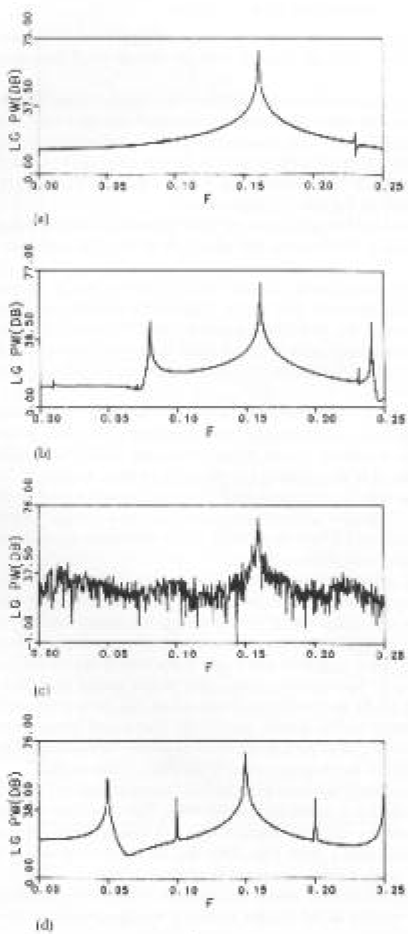


Fig. 14. (a)–(d) The phase portraits X(t + T) versus X(t) of the solutions of Eq. 13. Parameters are the same as the ones in Figure 13. Here, time interval dt = 0.05 and time delay T = 1. A typical strange attractor can be seen in (c).

Fig. 15. (a)–(d) The power spectra of the solutions of Fig. 13. The number of sampling points is 4,096. Only the lower quarter of the spectrum is displayed. The parameters are the same as in Figure 13. The highest peak in all plots is the fundamental frequency f_1. The second highest peak is the subharmonic frequency f_2, which can be seen in (b) and (d). A typical chaotic spectrum is shown in (c).



Phase transition and the pattern stability

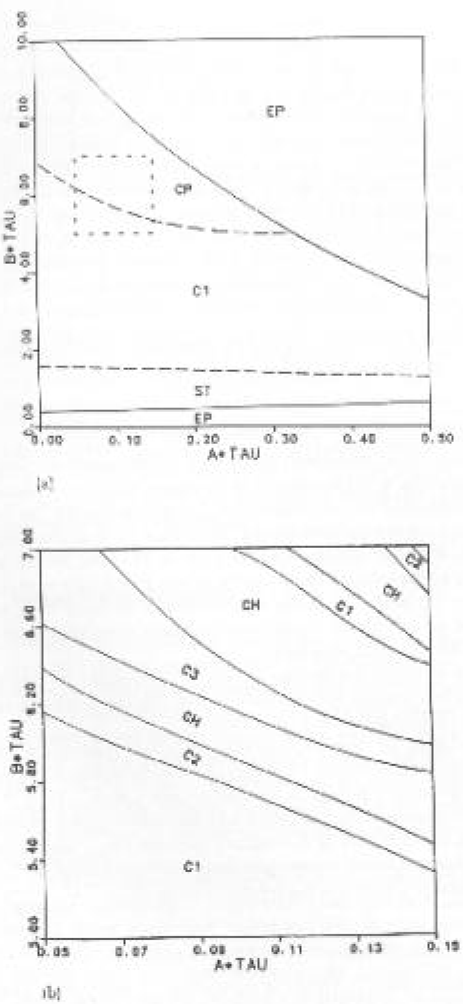
A more useful approach is to study the waveform of business cycles, since spectral analysis is difficult to apply with the few cycles of data available in economic time series.

The observed periodic repetition often consists of basic patterns with several shorter cycles. We define the number of shorter cycles in a basic wave pattern as the cycle number Ck. The basic pattern may have L large amplitude oscillations followed by S small amplitude oscillations. Each periodic state can be labeled the cycle number, Ck=L+S. For example, the periodic states in Figures 14a, 14b and 14d can be labeled C1, C2, and C3, respectively.

We should point out that the cycle number Ck is not necessarily equal to the period number P. For example, the wave form of P6 is C3, and those of P4 and P8 are belong to C2.

The phase diagram in terms of cycle number of the solutions is useful in characterizing economic long waves. Figures 16a and 16b display qualitatively the phase diagram of equation Eq. 13 in the parameter space. The broad diversity of dynamical behavior includes steady state ST, limit cycle or periodic motion C1, and explosive solution EP. The complex regime CP includes alternate periodic state (C1, C2, C3) and chaotic regime CH.

Fig. 16. The phase diagram of the numerical solutions of Eq. 13 in parameter space aτ and bτ. The dashed square area in (a) is enlarged in (b). Here, EP, ST, and CP represent explosive regime, steady state (after damped oscillation), and complex regime, respectively. CH is chaotic regime. C1, C2, and C3 are periodic patterns whose longer waves consist of one, two, or three shorter waves in turn.



When parameter values change within each region, the dynamic behavior is pattern-stable, because the dynamic mode occupies a finite area in the parameter space. The phase transition occurs when parameters cross the boundary between different phases. It is observable when the wave pattern changes.

The notation of cycle number Ck is introduced for possible application in analyzing long waves. An interesting feature of the model is that only three periodic patterns C1, C2 and C3 have been found. The model gives an simple explanation of multiperiodicity in business cycles.

It is speculated that no unique periodicity is involved in the business cycles. In addition to seasonal changes, several types of business cycles have been identified by economists [Van Duijn 1983]. The Kitchin cycles usually last 3-5 years; the Juglar cycles, 7-11 years; the Kuznets cycles, 15-25 years; and Kondratieff cycles, 45-60 years. Schumpeter suggested that these cycles were linked. Each longer wave may consists of two or three shorter cycles. This picture can be described by the periodic phase C2 or C3 in the CP regime of our model. The irregularity in long waves can also be explained by the chaotic regime CH. Our model gives a variety of possibilities of periodicity, multiperiodicity and irregularity in economic history, although our data only show the chaotic pattern in monetary movements.

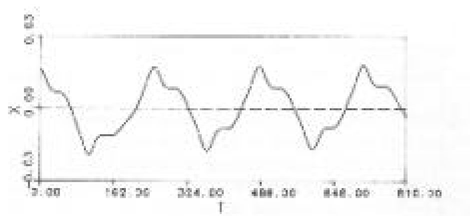
It is widely assumed that the long waves are caused by long lags, a belief coming from the linear paradigm [Rostow 1980]. This condition is not necessary in our model, because the dynamic behavior of Eq.13 depends both on $a\tau$ and $b\tau$. A strong overshooting plus a short time delay has the same effect as a weak control plus a long time delay, a similar point also made by Sterman [Sterman 1985].

This model is so simple and general, it could have applications beyond the monetary system in the market economy we discussed here. For example, the growth cycles and long waves caused by overshooting and time delay may also happen in centrally planned economies.

Simulating empirical cycles and forecasting basic trends

In comparing model-generated patterns with empirical data, we may confine our experiments to certain regions of the parameter space. For example, we can estimate the average period T from 4 times the decorrelation time T_d. The time delay τ in monetary control due to regulation lag and information lag is between 20 and 56 weeks [Gordon 1978]. If we estimate the time delay τ to be 39 weeks, we can simulate LD SSM2 time series by the solution shown in Figure 13c, by setting τ=39, a=0.00256, b=0.154, and σ=0.0125. The model results match well the average amplitude A_m, decorrelation time T_d, positive maximum Lyapunov exponent λ, and correlation dimension D of the empirical time series.

Fig. 17 The time path of medium-term growth cycles simulating the LD SSM2 time series in Figure 5a by means of Eq 12. Its long-term picture is the same as that in Figure 13c with a changed time scale.



The medium-term picture of simulated LD SSM2 in Figure 17 has well-behaved peaks and troughs with a stable period. We can hardly imagine that its long-term behavior is chaotic (see Figure 13c).

We tested the theoretical models with power spectra and autocorrelation analysis. The approximated period T of the chaotic solution can be estimated from T_d measured by 3-5 cycles. It is close to the fundamental period T_1 (=f_1^{-1}) determined by power spectra measured by 100 cycles with an error within 3 percent. For LD SSM2 time series, the difference of T_d measured between 10-15 years is less than 5 percent. We can obtain valuable information about the fundamental period T_1 without knowing the exact parameters of the deterministic model.

Implications for forecasting and control policy

We should point out that the word *chaos* is misleading. Chaotic motion has both regular and irregular characteristics. We prefer to refer to continuous deterministic chaos as *imperfect periodic motion*, which has a stable fundamental period but an irregular wave shape and a changing amplitude. Actually, we may often recover more information from chaotic motion than from random movements. For example, econometric models based on linear stochastic processes mainly explain the variance of the residuals. They offer little information about the trend and periods of business cycles beyond the short term. We suggest a new forecasting approach based on detecting strangeness of growth cycles. Although the long term prediction of the chaotic orbit is impossible from the view of nonlinear dynamics, a medium-term prediction of approximate period T can be made if we can identify strange attractors from the time series.

Let us discuss the meaning of the control parameters in Eq.12. When b=0, the monetary deviation from the natural rate will grow at a speed e^{at}. We define a characteristic doubling time t_a which measures the time needed to double the autonomous monetary expansion X(t) without control. Similarly, we can define a characteristic half time t_b, which measures the time needed to reduce the money supply to half its level when a=0 and X(t-τ) reaches the control target $X_m = \sigma/\ 2 = 1.4$ percent per year. The same is true for the contraction movements, since the feedback function G(x) is symmetric. Here t_a = 5.2 year and t_b=7.4 week for SSM2 in our simulation. We see that even modest time delay and overshooting may generate cycles and chaos.

For policy considerations, we suggest that the fluctuations in money supply can be moderated by reducing the time delay †t or control parameter b. We can set $7.3 < \tau < 29.3$ weeks while fixing a and b; or let $29.5 < t_b < 108.7$ weeks (when $1.51 > b > 0.41$). These figures give a qualitative picture of monetary target policy which seems reasonable for the real economy.

Summary and discussion

Empirical evidence of low-dimensional strange attractors is found in log linear detrended monetary aggregate data for the United States. These results are very encouraging, since new information is revealed about macroeconomic movements.

A differential-delay equation with only two parameters is suggested to describe monetary growth cycles. Self-generated periodic, multiperiodic, and chaotic behaviors are observed in the deterministic model. This model sheds light on the mechanism of business cycles and long waves : the nonlinearity and time delay in feedback control may cause complex behavior. Although our model is simple and exploratory, it has enabled us to simulate the wave pattern and low dimensionality of monetary growth cycles.

We do not deny the complexity of social phenomena and the usefulness of disaggregated approaches in econometrics and system dynamics. Low-dimensional economic chaos is not only useful but also testable in economic studies. It can be understood through the experience of physicists. It is often convenient to introduce projection operators which decompose the system into one low-dimensional space, whose movements can be effectively simplified, and one orthogonal to it [Prigogine 1980]. In practice, the right choice of the projection operator can only be made by empirical tests. Our work, together with previous efforts in study of complex systems, strongly supports the hope that social phenomena can be quantitatively described by simple mathematical models in some aspects. The key issues are which pertinent variable to observed and what can we learned from the model.

Three problems remain to be solved for future studies of economic chaos.

· The main obstacle in empirical analysis arises from limited data sources in economics. In order to facilitate the testing of deterministic chaos and to improve our understanding of modern economies, it is worthwhile to develop numerical algorithms that work with moderate data sets, as well as to expand the data base of economic statistics.

· The second question is how to determine the reference system. In our numerical experiment, the starting and ending periods of the observations were arbitrarily dictated by the available data. We do not know if the natural rate of growth is a constant or changing over time. Perhaps this problem can be solved by future testing on longer period combined with the effort of a historian helping to identify turning points of economic history. It is advantageous for nonlinear dynamics to introduce a time arrow or a historical perspective in analyzing complex systems [Prigogine 1980].

· The third issue is how to estimate parameters from empirical data. We should point out that the solutions of a nonlinear delay-differential equation may not be approximated by one- or two-dimensional discrete models in fitting the empirical data. We should be cautious in applying conventional technique of econometrics to chaos models.

Exploring economic chaos opens a new way to understand human behavior and social evolution. The studies of nonequilibrium and nonlinear phenomena have not only changed the techniques we use but also the ways in which we think [Prigogine and Stengers 1984].

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Figure captions

Fig. 1. Comparison of the time series of model solutions. Their time units are arbitrary. (a) AR(2) linear stochastic model. (b) Discrete logistic chaos generated by mapping X(t+1) = 4X(t)[1-X(t)]. (c) Rossler model of spiral chaos with time interval dt = 0.05.

Fig. 2. Comparison of the phase portraits of model solutions.N=1000. (a) AR(2) model with T=20. (b) Logistic chaos with T=1. (c) Rossler model with T=1 and dt =0.05.

Fig. 3. Comparison of the autocorrelations of the three model solutions with 1000 data points. The time units are same as in Figure 1.

Fig. 4. The exponential growth trends in time series of monetary aggregates :Official simple-sum index SSM2 and Divisia index DDM2 (January 1969 - July 1984). The time unit is one week.

Fig. 5. Comparison of the detrended weekly time series SSM2. (a) Symmetric LD SSM2: the log linear detrended SSM2 with a natural growth rate of 4 percent per year. (b) Asymmetric FD SSM2 : the logarithmic first differences of SSM2.

Fig. 6. Comparison of the phase portraits of empirical time series. Time delay T=20. (a) LD SSM2 time series. Time unit is one week. N=807 points. (b) IBM daily common stock returns. The time interval is one day. N=1000 points, beginning on July 2, 1962.

Fig. 7. Comparison of autocorrelation functions; AC(I) plotted against I. There are three time series: LD SSM2, LD DDM2 and IBM daily stock returns, each in their original time units. N=807.

Fig. 8. An artist's sketch of the Wolf algorithm. The lower line, y(t), is the reference orbit. The upper broken line, z(t), starting in the neighborhood of $y(t_0)$, is traced to calculate the divergence of the lines. The points $z_0(t_1)$, $z_1(t_2)$, . . ., are replaced by new nearest neighboring points $z_1(t_1)$, $z_2(t_2)$ after an evolution time EVOLV for numerical calculation.

Fig. 9. The maximum Lyapunov exponents of log linear detrended monetary aggregates LD SSM2 and LD DDM2. The maximum Lyapunov exponents of monetary aggregates plotted against the evolution time EVOLV, calculated in phase space with time delay T=5 weeks and embedding dimension m=5. The unit is bit per week. The evolution time EVOLV in calculating numerical Lyapunov exponent varies from 15 to 180 weeks at 15-week intervals.

Fig. 10. The Grassberger-Procaccia plots for calculating correlation dimension of LD SSM2 time series with time delay T=5. The embedding dimension m=2, . . . , 6, is taken as a parameter. (a) Plots of $\ln_2 C_m(R)$ versus $\ln_2 R$, The plots rotate downwards and to the right as m increases. (b) Plot of the slopes of the curves in (a) against the $\ln_2 R$. The linear region of the curves in (a) can be identified from the plateau region in (b). The correlation dimension is equal to the saturated slope 1.5 for LD SSM2 measured from the plateau region with m=5.

Fig. 11. (a) and (b) The Grassberger-Procaccia plots for calculating correlation dimension of LD DDM2. Its correlation dimension is 1.3.

Fig. 12. The flow diagram for a delayed feedback system of economic growth.

Fig. 13. The time sequences of the numerical solutions of Eq.13. The parameters were fixed at a=0.1 and τ=1 while changing the parameter b. (a) Period-1 solution P1 (limit cycle) with b=5.7. Its cycle number is C1. (b) Period-2 solution P2 (C2) with b=5.8. (c) Chaotic solution CH with b=6.0. (d) Period-3 solution P3 (C3) with b=6.3.

Fig. 14. (a)-(d) The phase portraits X(t+T) versus X(t) of the solutions of Eq.13. Parameters are the same as the ones in Figure 13. Here, time interval dt =0.05 and time delay T=1. A typical strange attractor can be seen in (c).

Fig. 15. (a)-(d) The power spectra of the solutions of Eq.13. The number of sampling points is 4096. Only the lower quarter of the spectrum is displayed. The parameters are same as in Figure 13. The highest peak in all plots is the

fundamental frequency f_1. The second highest peak is the subharmonic frequency f_2, which can be seen in (b) and (d). A typical chaotic spectrum is shown in (c).

Fig. 16. The phase diagram of numerical solutions of Eq.16 in parameter space $a\tau$ and $b\tau$. The dashed area in (a) is enlarged in (b). Here, EP, ST, and CP represent explosive regime, steady state (after damped oscillation), and complex regime, respectively. CH is chaotic regime. C1, C2, and C3 are periodic patterns, whose longer wave consists of one, two, or three shorter waves in turn.

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