Mathematical Representation in Economics and Finance: Philosophical Preference, Mathematical Simplicity, and Empirical Relevance

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Abstract As Keynes pointed out, classical economics was similar to Euclidean geometry, but the reality is non-Euclidean. Now we have abundant evidence that market movements are nonlinear, non-equilibrium, and economic behavior is collective in nature. But mainstream economics and econometrics are still dominated by linear, equilibrium models of representative agent. A critical issue in economics is the selection criteria among competing math models. Economists may choose the preferred math representation by philosophical preference; or by mathematical beauty or computational simplicity. From historical lessons in physics, we choose the proper math by its empirical relevance, even at the costs of increasing mathematical dimensionality and computational complexity. Math representations can be judged by empirical features and historical implications. Recent historical events of financial crisis reveal the comparative advantage of the advanced math representation. Technology progress facilitates future advancements in mathematical representation and philosophical change in economic thinking.

I. Introduction: What's Wrong with Economic Math?

There is a quiet revolution in scientific paradigm that began in 1970s [1] that has increasing impact to mathematical economics since 1980s [2]. Advances in nonlinear dynamics and non-equilibrium physics fundamentally change our views from the origin of life to the evolution of universe. However, mainstream economics is reluctant to adopt new complexity science and its application to economics. Before and after the 2008 financial crisis, there is a strong criticism of excess mathematical economics and its effort in imitating

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physics [3, 4, 5]. However, few pointed out what's wrong with economic math. Krugman blamed economics for "mistaking beauty for truth," [6]. However; he did not elaborate how to judge beauty and truth in economics.

As trained as a physicist, a more fundamental question is: Why does economics need math? There are two possible answers: First, using math as a tool for discovering regularities from large numbers of numerical data, just like physicists or physicians do when analyzing experimental data. Second, using math as a language to express a concept or belief, so that people may accept economics as a branch of science in the modern era. This motivation is visible in neoclassical economics.

And then, we have another question: how to select a preferred model among several competing representations based on the same data set? We have two possible choices: we may select a math model based on its empirical relevance or mathematical beauty. This is the main issue discussed in this article.

Introducing math into economics is a development process. Before the Great Depression in 1930s, economic debate was mainly contested by historical stories and philosophical arguments. After the Great Depression, an increasing number of economic data has been collected by governments and corporations. Rising demand in data analyses stimulates rapid application of statistics and math in economic studies. The IS-LM model in neoclassical economics became a professional language in policy debate on fiscal and monetary policy. Furthermore, economic math began to dominate academia and universities since 1950s. The equilibrium school succeeded in promoting their belief in self-stabilizing market mainly by intellectual power in economic math. Their weapon in math representation is the unique equilibrium in linear demand and supply curves. In contrast, the Austrian school emphasized the organic nature of economic systems. They used philosophical arguments against neoclassical models. Since 1980s, the evolutionary school has been integrating nonlinear dynamics with evolutionary mechanisms.

Now we face a methodological and philosophical challenge: which school is better at understanding contemporary issues such as business cycles, economic crisis, economic growth, and stabilizing policy. And which mathematical representation is a better tool in quantitative research in economics and finance.

In this article, we will demonstrate different math representations and competing economic thoughts, and discuss related issues in economic philosophy.

II. Mathematical Representations of Competing Economic Thoughts

The central issue in economics and finance is the cause of market fluctuations and government policy in dealing with business cycles. The equilibrium school believes that market movements are self-stabilizing; therefore the laissez-fare policy is the best option. The disequilibrium school may use fiscal and monetary policy when market is out of equilibrium in a short-term. The non-equilibrium school studies more complex situations including economic chaos and multiple regimes. Math models are widely used in economic debate since 1930s.

Competing economic math is closely associated with competing schools in economics. We may classify them into three groups: the equilibrium school led by neo-classical economics, the disequilibrium school led by Keynesian, and the complexity school led by evolutionary economics. From mathematical perspective, both equilibrium and disequilibrium school are using linear models and static statistics, while complexity school is developing nonlinear and non-equilibrium approach. We will demonstrate how economic math play a key role in economic thinking and policy debate.

(2.1) Single and Multiple Equilibriums in Linear and Nonlinear Demand Supply Curve

Neoclassical economics characterizes a self-stabilizing market by a single equilibrium state that is the cross point of the linear demand and supply curves (see Fig. 1a). In contrast, disequilibrium school describes market instability by multiple equilibriums, which are cross points of nonlinear demand and supply curves [7]. Their policy implications are quite explicit. Government interference is no need for single stable equilibrium, but crucial for avoiding bad equilibrium under multiple equilibrium situation.
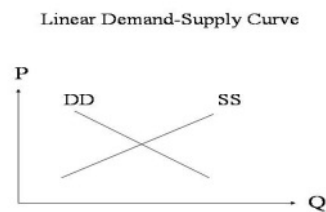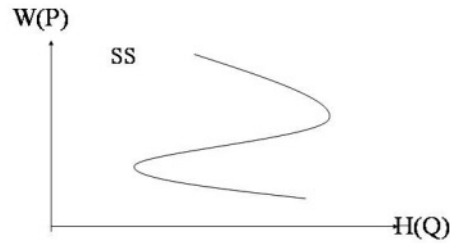

(a)

Nonlinear Demand Curve (Becker)
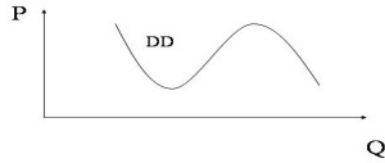
(b)

Nonlinear Supply Curve (Stiglitz)

(c)

Fig. 1. Linear and Nonlinear Demand Supply Curves in Micro. **a**. Linear demand and supply with single equilibrium.
b. Nonlinear S-shaped demand with social interaction.
c. Nonlinear Z-shaped labor supply with subsistence and pleasure needs.

Becker realized the possibility of S-shaped demand curve when social interaction plays an important role [8]. The herd behavior is observed from restaurant selection and market fads. Z-shaped labor supply curve implies increasing labor supply at low wage for survival needs, but reducing work at high wage for pleasure [9].

(2.2) Large Deviation and Unimodular Distribution

How to understand the degree of market fluctuation? The probability distribution provides a useful tool in studying stochastic mechanism. The simplest distribution is the unimodular distribution with a single peak. The degree of market fluctuation is measured against the Gaussian distribution with finite mean and variance that is widely used in econometrics [10] and capital asset pricing model in finance theory [11]. If a random variable follows a Gaussian distribution, and its standard deviation is σ. The probability of larger

than 3σ deviation from the mean is 0.3%, and 5σ deviation of only 0.00006%.

However, we often observe large market fluctuations or even crisis. Historical examples include the Black Monday in Oct. 19, 1987, Dot-com bubble in 1997-2000, and 2008 financial crisis. One possible explanation is a fat-tail distribution with single peak [12], such as the Cauchy distribution with infinite variance in probability theory. A special case is the power law studied in econophysics [13].

Both Gaussian and fat-tail belongs to the unimodular distribution with single peak. Their distributions are shown in Fig.2.
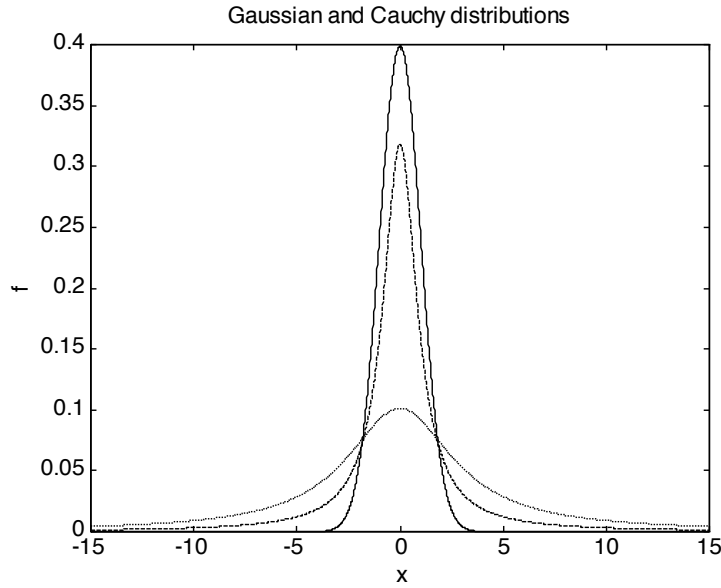
Fig. 2. Gaussian and Cauchy Distribution with One Peak.
The Gaussian distribution with zero mean and unit variance, N(0,1), is the tallest in solid line. The Cauchy distribution with single peak but varying height is shown here for comparison. Cauchy(0,1) distribution is in the middle in dashed line, and Cauchy(0,π) is the lowest and fattest distribution in dotted line. Clearly, fat-tail distribution has larger probability of large deviation.

(2.3) Bi-modular Distribution and Regime Switch in Social Psychology

What is the cause of large market fluctuation? Some economists blame irrationality behind the fat-tail distribution. Some economists observed that social psychology might create market fad and panic, which can be modeled by collective behavior in statistical mechanics. For example, the bi-modular distribution was discovered from empirical data in option prices [14]. One possible mechanism of polarized behavior is collective action studied in physics and social psychology [15]. Sudden

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regime switch or phase transition may occur between uni-modular and bi-modular distribution when field parameter changes across some threshold.

Here, we discuss two possible models in statistical mechanics. One is the Ising model of ferromagnetism; and another is population model of social interaction.

(2.3.1) Ising Model of Social Behavior in Equilibrium Statistical Mechanics

The Ising model in equilibrium statistical mechanics was borrowed to study social psychology. Its phase transition from uni-modular to bi-modular distribution describes statistical features when a stable society turns into a divided society, like recent events in Ukraine and the Middle East [16]. Fig.3 shows the three regimes in Ising model of public opinion.
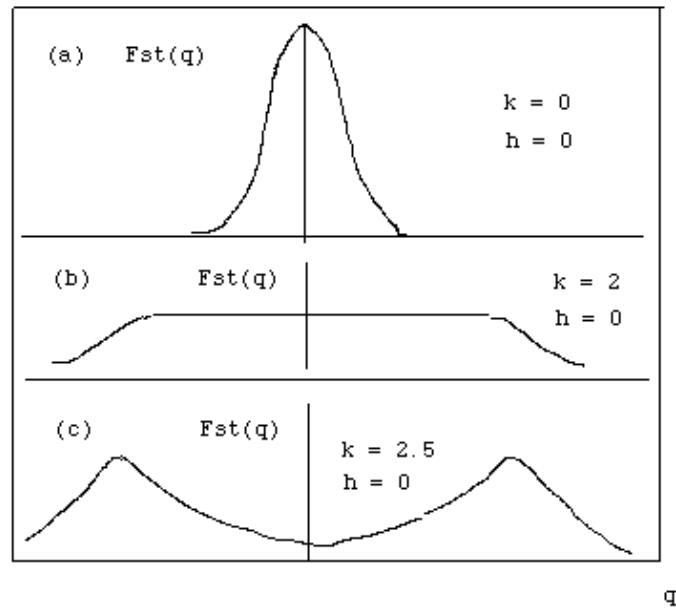

Fig.3. The steady state of probability distribution function in the Ising Model of Collective Behavior with h=0 (without central propaganda field). **a**. Uni-modular distribution with low social stress (k=0). Moderate stable behavior with weak interaction and high social temperature. **b.** Marginal distribution at the phase transition with medium social stress (k=2). Behavioral phase transition occurs between stable and unstable society induced by collective behavior. **c.** Bi-modular distribution with high social stress (k=2.5). The society splits into two opposing groups under low social temperature and strong social interactions in unstable society.

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(2.3.2) Population Model with Social Interaction in Non-Equilibrium Statistical Mechanics

The problem of the Ising model is that its key parameter, the social temperature, has no operational definition in social system. A better alternative parameter is the intensity of social interaction in collective action. The three regimes of the probability distribution can be obtained by the master equation [17]. The U-shaped distribution in Fig. 4c is similar to the bi-modular distribution in Fig. 3c.
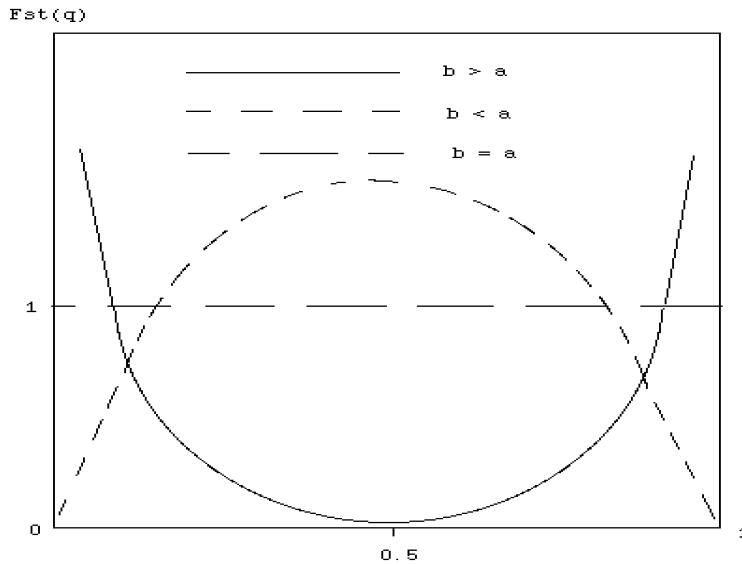

Fig.4.The steady state of probability distribution function in socio-psychological model of collective choice. Here, "a" is the independent parameter; "b" is the interaction parameter.
a. Centered distribution with b < a (denoted by short dashed curve). It happens when independent decision rooted in individualistic orientation overcomes social pressure through mutual communication.
b. Horizontal flat distribution with b = a (denoted by long dashed line.) Marginal case when individualistic orientation balances the social pressure.
c. Polarized distribution with b > a (denoted by solid line). It occurs when social pressure through mutual communication is stronger than independent judgment.

(2.4) Business Cycles: White Noise, Persistent Cycles and Color Chaos

A more difficult issue in business cycle theory is how to explain the recurrent feature of business cycles that is widely observed from macro and financial indexes. The problem is: business cycles are not

strictly periodic and not truly random. Their correlations are not short like random walk and have multiple frequencies that changing over time. Therefore, all kinds of math models are tried in business cycle theory, including deterministic, stochastic, linear and nonlinear models. We mainly discuss three types of economic models in terms of their base function, including white noise with short correlations, persistent cycles with long correlations, and color chaos model with erratic amplitude and narrow frequency band like biological clock.

Deterministic models are used by Keynesian economists for endogenous mechanism of business cycles, such as the case of the accelerator-multiplier model [18].

The stochastic models are used by the Frisch model of noise-driven cycles that attributes external shocks as the driving force of business fluctuations [19]. Since 1980s, the discovery of economic chaos [20] and the application of statistical mechanics [21] provide more advanced models for describing business cycles. We will show their main features in mathematical representation.

(2.4.1) Linear Model of Harmonic Cycle and White Noise

Linear harmonic cycles with unique frequency are introduced in business cycle theory [22]. The auto-correlations from harmonic cycle and white noise are shown in Fig. 5.
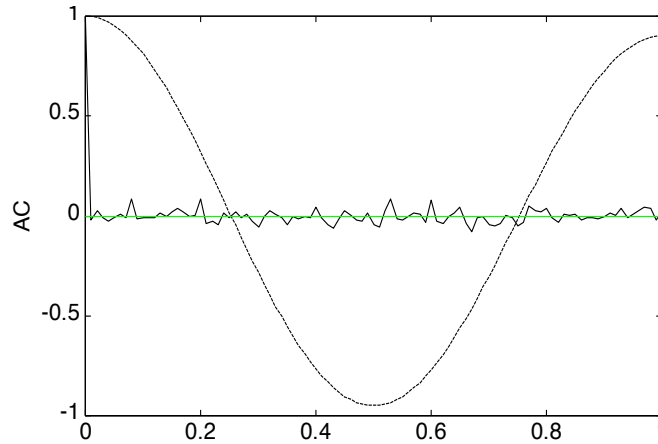
Fig.5. Numerical autocorrelations from time series generated by random noise and harmonic wave. The solid line is white noise. The broken line is a sine wave with period P = 1.

Auto-correlation function from harmonic cycles is a cosine wave. The amplitude of cosine wave is slightly decayed because of limited data points in numerical experiment. Auto-correlations from a random series are an erratic series with rapid decade from one to residual fluctuations in

numerical calculation. The auto-regressive (AR) model in discrete time is a combination of white noise term for simulating short-term auto-correlations from empirical data [23].

(2.4.2) Nonlinear Model of White and Color Chaos
The deterministic model of chaos can be classified into white chaos and color chaos.

White chaos is generated by nonlinear difference equation in discrete-time, such as one-dimensional logistic map [24, 25] and two-dimensional Henon map [26, 27]. Its autocorrelations and power spectra look like white noise. Its correlation dimension can be less than one. White noise model is simple in mathematical analysis but rarely used in empirical analysis, since it needs intrinsic time unit.

Color chaos is generated by nonlinear differential equations in continuous-time, such as three-dimensional Lorenz model [28] and one-dimensional model with delay-differential model in biology [29] and economics [30]. Its autocorrelations looks like a decayed cosine wave, and its power spectra seem a combination of harmonic cycles and white noise. The correlation dimension is between one and two for 3D differential equations, and varying for delay-differential equation. We will show later that only color chaos is observed from empirical economic indexes.

The most visible feature of deterministic chaos is their phase portrait. The typical feature of color chaos is its spiral pattern in its phase portrait in Fig. 6.
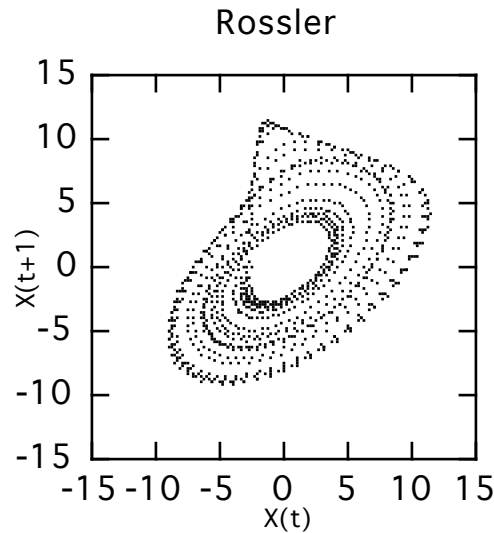

Fig. 6. The phase portrait of the Rössler color chaos in continuous time [31].

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(2.4.3) Fragile and Resilient Cycles in Linear and Nonlinear Oscillator

History shows the remarkable resilience of a market that experienced a series of wars and crises. The related issue is why the economy can recover from severe damage and out of equilibrium? Mathematically speaking, we may exam the regime stability under parameter change.

One major weakness of the linear oscillator model is that the regime of periodic cycle is fragile or marginally stable under changing parameter. Only nonlinear oscillator model is capable of generating resilient cycles within a finite area under changing parameters. The typical example of linear models is the Samuelson model of multiplier-accelerator [32] In Fig.11a, periodic solution PO only exists along the borderline between damped oscillation (DO) and explosive oscillation (EO).
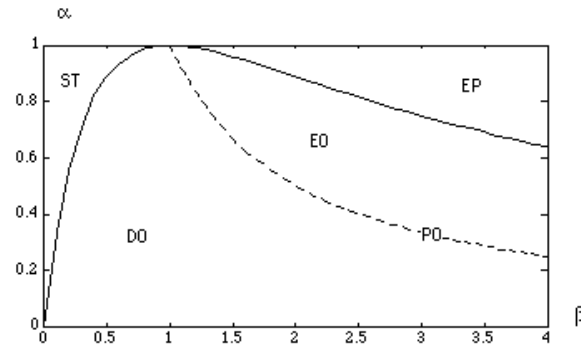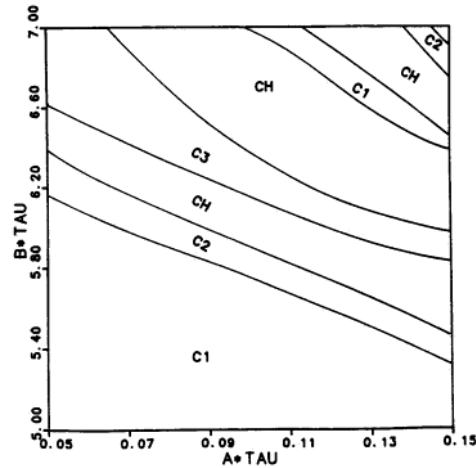

a. Stability pattern of Samuelson model in parameter space (1939). Here, ST denotes the steady state; DO, damped oscillation; EO, explosive oscillation; EP, explosive solution; PO, linear periodic oscillation.
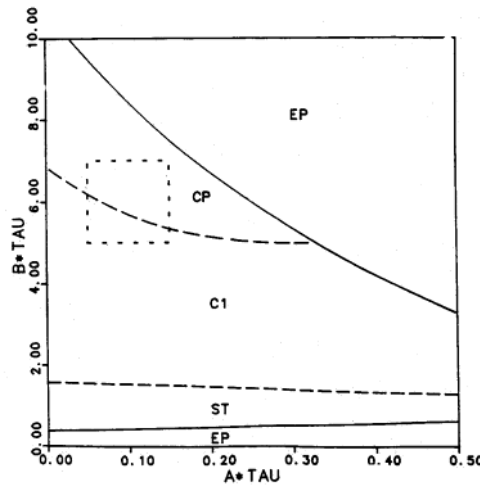
b. Parameter space for soft-bouncing oscillator. ST denotes the steady state. CP is the complex regime including multi-periodic states C1, C2,

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C3, etc.

(7c) The expanded regime in (11b). C1, C2, C3 are limit cycles of period one, period two, and period three respectively; CH, the chaos mode in continuous time.

Fig. 7. Structural stability in parameter space. (7a) Periodic solution PO is only marginally stable at the borderline. (7b) Complex and chaotic regime is structurally stable within the area of CP. The complex regime CP in (7b) is enlarged in CH in (7c) that consists of alternative zones of limit cycles and chaos.

Fig. 7b and Fig.7c is the parameter space of the nonlinear dynamic model of soft-bouncing oscillator [33]. Its nonlinear periodic solution P1 (period one, the limit cycle), P2 (period 2), P3 (period3), and CH (chaotic solution) are structurally stable when parameter changes are within the CP area. The phase transition or regime switch occurs when parameter change is cross the boundary of CH or CP.

Linear stochastic models have similar problem like linear deterministic models. For example, the so-called unit root solution occurs only at the borderline of the unit root [34]. If a small parameter change leads to cross the unit circle, the stochastic solution will fall into damped (inside the unit circle) or explosive (outside the unit circle) solution.

(2.5) Logistic Wavelet, Metabolic Growth and Schumpeter's Creative Destruction

Any living system has a life cycle that can be described by a wavelet with finite life, which is a better math representation than harmonic wave with infinite life or white noise with zero life.

The origin of wavelet representation in economics can be traced to resource limit in market-share competition. The ecological constraint is the major source of economic nonlinearity [35]. The simplest model of resource limit in theoretical ecology is the logistic growth. Species competition in ecology can be used as technology competition model. A well-known example is the Lotka-Volterra model with two species. Fig.8 shows the outcome of two competing species with different carrying capacity.
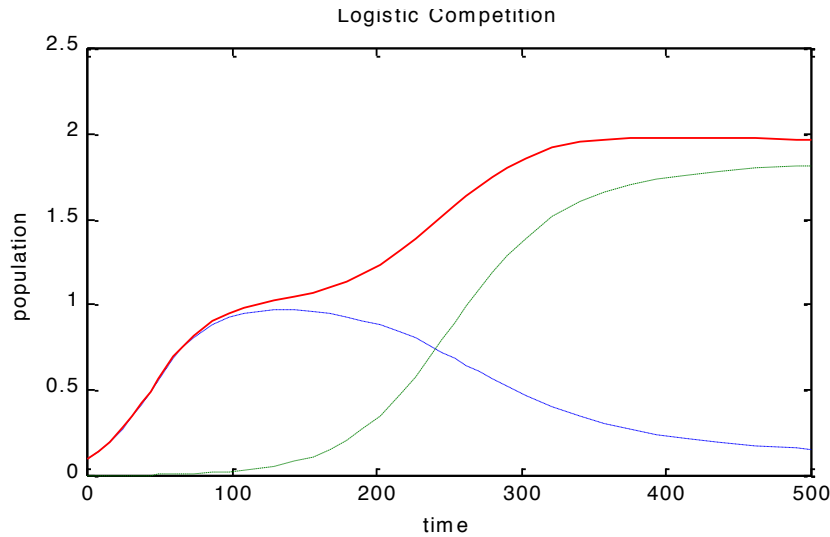
Figure 8. Staged economic growth is characterized by the envelop of the sum of competing logistic wavelets that mimics the observed pattern of a time series from macroeconomic indexes.

The logistic wavelet with a finite life is a simple nonlinear representation for technology life cycles. Schumpeter's long waves and creative destruction can be described by a sequence of logistic wavelets in a technology competition model [36].

A numerical solution of competition equation is shown in Fig.8. Without technology 2's competition, the growth path of technology (species) 1 (on the left) would be a S-shaped logistic curve (on the right). However, the realized output of technology 1 resulting from competition with technology (species) 2 looks like an asymmetric bell curve. We call it the logistic wavelet, which is a result from the competition of new technology. Clearly, logistic wavelet is a better representation for technology progress than random shocks in RBC (real business cycle)

literature, since the technology competition model described Adam Smith's idea of market-share competition [37]. Schumpeter's creative destruction can be explained by over-capacity under technology competition, since old and new technology market-share are below their full capacity. The so-called "insufficient aggregate demand" in Keynesian economics could be resulted from "excess capacity" caused by technology competition for market share.

(2.6) Persistent Fluctuation and Population Dynamics

A new measurement of relative deviation (RD) reveals the relation between persistent fluctuations and population dynamics. Economic statistics often measure mean and variance separately. For positive variables, such as price and population, RD is the ratio of standard deviation to its mean [38]. Rd is an important indicator for persistent fluctuations (see Fig.9.)
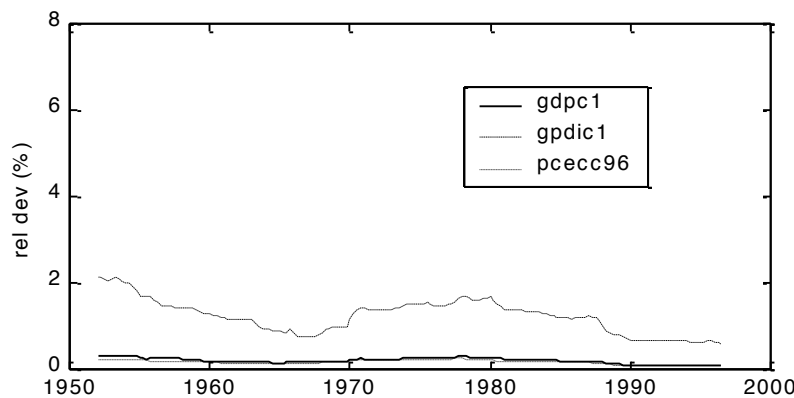
Fig. 9. The RDs of GDPC1 (US real GDP), GPDIC1 (real investment), and PCECC96 (real consumption) for the US quarterly series (1947-2001). N=220. Moving time window is 10 years. Displayed patterns were observed through the HP filter.

Fig.9 shows the relative deviation (RD) from three US macro indexes, including real GDP, real investment, and real consumption. All three curves have no trend toward infinite explosion or zero convergence, only oscillate within finite range. We may ask what kind of stochastic models has the feature of persistent fluctuations.

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Three stochastic models are used in economics and finance theory. The random walk model [39] and the Brownian motion model [40] belong to the representative agent model, since they describe the trajectory of only one particle. The birth-death process is also introduced in physics and finance [41]. Their statistics in time is given in Table 1.

Table 1. Statistics for three linear stochastic models

Model	Brownian Motion	Birth-Death	Random-Walk
Mean	$\sim \exp(rt)$	$\sim \exp(rt)$	$\sim t$
Variance	$\sim \exp(2rt)\{e^{\sigma^2 t}-1\}$	$\sim e^{rt}(e^{rt}-1)$	$\sim t$
RD	$\sim e^{\frac{\sigma^2}{2}t}\sqrt{(1-e^{-t\sigma^2})}$	$\sim \dfrac{1}{\sqrt{N_0}}$	$\sim \dfrac{1}{\sqrt{t}}$

In this table, the first moment is the mean, the second moment is variance, RD is the ratio of the standard deviation (the square root of variance) to its mean. Here, N_0 is the size of initial population of particles in the birth-death process and r>0 for economic growth.

From Table 1, we can see that the RD of Brownian motion is explosive in time, RD of random walk is damping. Only the RD of the birth-death process tends to a constant. This is a critical criterion in choosing proper math model in finance theory [42].

Clearly, the Brownian motion and random walk model are not proper for modeling macro dynamics in the long-term. Only the birth-death process is qualified for both macro and financial dynamics with persistent RD. We found that Brownian motion model is explosive in 2005. We speculated that the collapse of the derivative market in 2008 financial crisis might be caused by the theoretical flaw in option pricing model, since interest swap pricing model is also based on the Brownian motion. We suggest that option pricing should be based on the birth-death process [43, 44].

(2.5) The Frisch Model of Noise-Driven Cycles: a Math Delusion for Self-Stabilizing Market or an Economic Fallacy as a Perpetual Motion Machine?

The Frisch model of noise-driven cycles plays a central role in equilibrium theory of business cycle and econometrics. In mathematical construction, it is a mixed model of damped harmonic cycles and persistent white noise [45]. A harmonic oscillator with friction would automatically tend to stop. This dynamic feature is used for characterizing a self-stabilizing market. The problem is how to explain persistence of business cycles in history. Frisch suggested that harmonic cycles could be maintained by persistent shocks. Frisch made the claim

during the Great Depression in 1933. If the Frisch model is true, the Frisch model will save the liberal belief in a self-stabilizing market, since persistent business cycles are not generated by internal market stability but external random shocks. Frisch shared the first Nobel Prize in economics for this model in 1969. The noise-driven model is also behind the works by Lucas and RBC school.

However, physicists already knew before Frisch that thesedumped harmonic cycles could not be kept alive by random shocks [46], since its amplitude would decay exponentially [47]. We calibrate the Frisch model by the US data. We found that American business cycles would only last about 4 to 10 years [48]. In fact, the NBER recorded US business cycles since 1854, more than 160 years ago.

The Frisch conjecture implies a perpetual motion machine of the second kind. It is a working machine powered by random thermal fluctuations, which is a heat engine with single temperature source. This engine could not do any work, since any heat engine must burn fuel at high temperature and release waste heat at the low temperature according to the second law of thermodynamics.

Frisch claimed that he had already solved the analytical problem and that this would soon be published. His promised paper was advertised three times under the category "papers to appear in early issues" in 1933, but it never appeared in *Econometrica*, where Frisch served as the editor. Frisch did not mention a word about his prize-winning model in his Nobel speech in 1969 (Frisch 1981). Obviously, Frisch quietly abandoned his model since 1934, but never admitted his mistake in public. This story should be a wake-up call to equilibrium economics: there is only one step between economic belief and mathematical delusion.

III. Empirical Tests of Competing Economic Math

Empirical tests of competing math models in economics is more difficult than that in physics. There are two fundamental problems to be solved in conducting numerical tests of competing economic models.

First, many economic time series have a growing trend, since economy is an open system with increasing energy consumption. However, existing models of business cycles are stationary models. We need a proper mapping to transform non-stationary time series into stationary series, which is a similar issue in finding a preferred observation reference in planet motion. This is the Copernicus problem in economics.

Second, economic data is raw data that contain strong noise, which is more difficult for signal processing comparing to low noise data from controlled experiments in natural science. Conventional

band-pass filter is not sufficient to separate signal from noise. We need more advanced tools in analyzing non-stationary time series.

In this section, we will introduce two math tools in solving these two problems: the HP filter for defining smooth trend series and the time-frequency analysis in time-frequency two-dimensional space.

## (3.1) Observation Reference and the Copernicus Problem in Economics

Physics revolution started with the Copernicus, who changed the geocentric system by Ptolemy to heliocentric system for studies of planet motion. Economic analysis faces a similar choice of preferred observation reference.

Competing economic schools differ in their scope of time windows in analyzing economic time series.

Equilibrium school and econometrics believe that market is capable in self-adjustment towards equilibrium, so that they choose a short-term time window to obtain a random image of market movements. Their tool is calculating the rate of changes within a time unit from a growing time series, such as the daily or yearly returns of stock prices or GDP series. Mathematically speaking, it is equivalent to the first difference of a logarithmic time series, which is called the FD (first differencing) filter in econometrics. The FD de-trended time series look random. This is the empirical foundation of the so-called efficient market. An alternative mapping is taking a log-linear trend, which depends on the end points or the length of time series. The HP (Hodrick-Prescott) filter is a compromise between the FD and the log-linear detrending (LLD) filter. The HP filter defines a nonlinear smooth trend, which corresponds to a medium time window in the typical length of NBER business cycles about 20 to 10 years [49]. The HP filter was discussed by Von Neumann in analyzing time series with changing mean [50] We found out that the FD series look random but the HP cycles reveal nonlinear patterns and varying frequency. We may compare their results in Fig. 10.

Trends of FSPCOMln Index
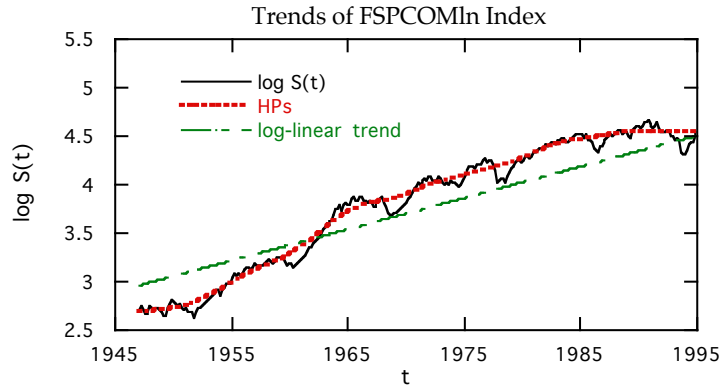
(a). HP trend and LLD (log-linear) trend for X(t) {=log S(t)}. LLDc cycles are residuals from log-linear trend.

Cycles of FSPCOMln Index
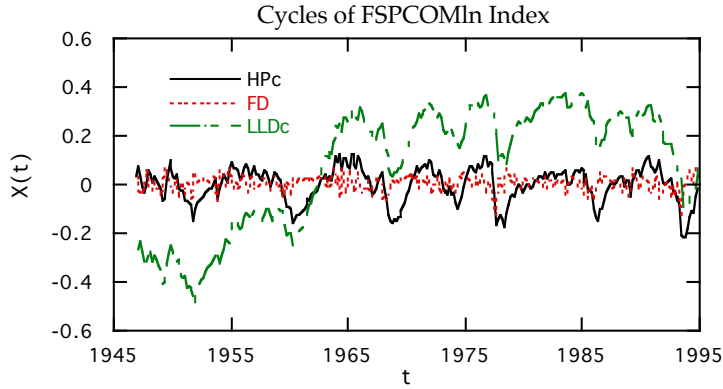
(b). Cycles from competing detrending filters.

Autocorrelations of FSPCOMln Cycles
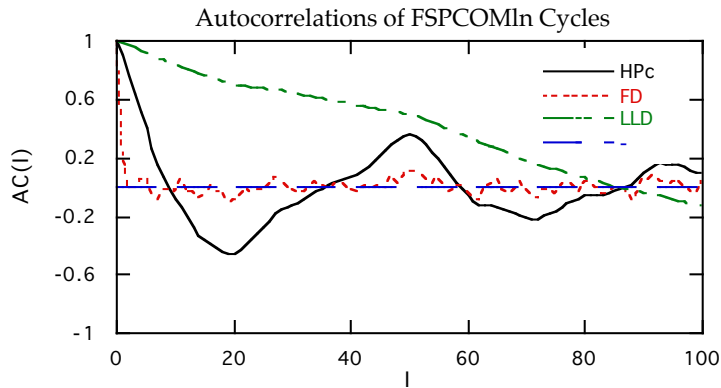
(c). Autocorrelations of detrended series. The length of correlations is the longest from LLD, medium from HP, and shortest from FD.

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Fig. 10. Varying correlated length from FD, HP, and LLD detrending filtered from the logarithmic FSPCOM (S&P 500) monthly series (1947-92). N=552.

From the length of the first zero autocorrelation in Fig. 10c, we may estimate the length of the period from detrended series. FD cycle is 0.7 year, HPc 3 years, LLD 29 years. Clearly, HP cycles are the best indicator to characterize the US business cycles.

(3.2) Filter Design in Signal Processing

Signal processing is a technology field, which rapidly developed since 1950s. Scientists make a great effort to reduce noise level and enhance signal for getting better information. Band-pass filter is widely used in empirical analysis. Only one discipline made an odd choice: The FD filter in econometrics is used for amplifying high frequency noise. Its frequency response function reveals its secret in Fig. 11 [51].
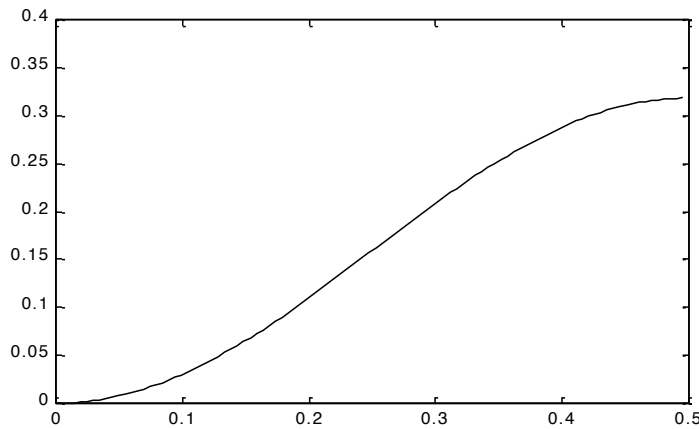

Fig.11. Frequency response function for the FD filter.

Here, X(t) = FD[S(t)]=S(t+1)-S(t), the horizontal axis

is the frequency range from zero to 0.5.

In other words, econometricians use a whitening looking glass in empirical observation. A colorful landscape in the real world would look like randomly distributed white spots through the FD filter. Why?

Because equilibrium economics asserts that efficient market should behave like white noise. The FD filter plays a role of illusion creating, rather than signal extracting.

In contrast, we developed a more advanced filter in two-dimensional time-frequency Gabor space for analyzing non-stationary time series [52]. According to quantum mechanics, the uncertainty principle in time and frequency indicates two methods to minimize uncertainty in signal processing when the envelope of harmonic cycles is Gaussian, which is called the Wigner transform. We can project the original time series onto a two-dimensional time-frequency Gabor discrete lattice space (see Fig.9a), and using Wigner function as base function in every grid (Fig. 9b). Noise component in Gabor space can be easily separated like mountain (signal) and sea (noise) in Fig.9c. The filtered time series are shown in Fig.13.

The uncertainty principle in time and frequency is the very foundation of signal processing:

$$\Delta f \, \Delta t \geq \frac{1}{4\pi} \tag{1}$$

Application of Joint Time-Frequency Analysis to ISAR

DFT-Based ISAR

JTFA-Based ISAR Image of a MIG-25
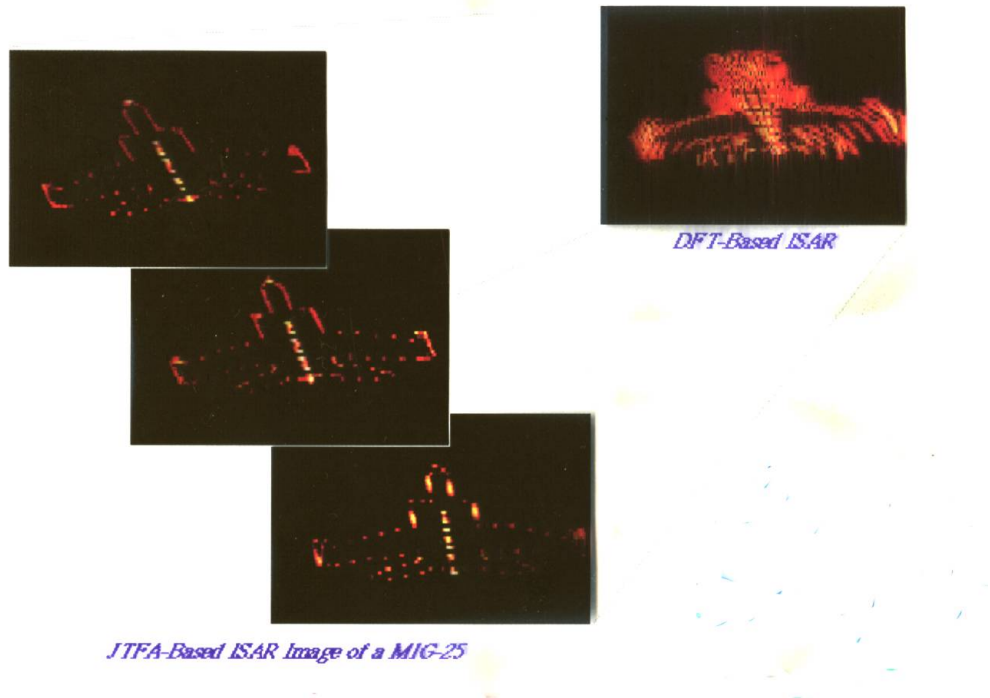
(a) Non-stationary (left picture) vs. stationary (right picture) time series analysis of a moving object (MIG-25 fighter plane).
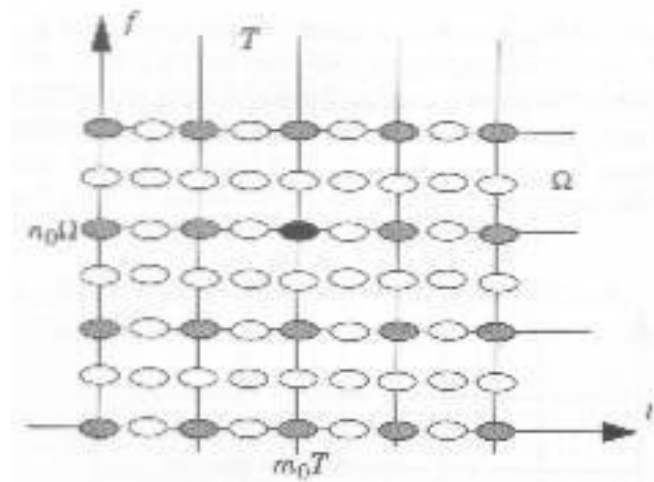
(b) Gabor Lattice Space in Time and Frequency
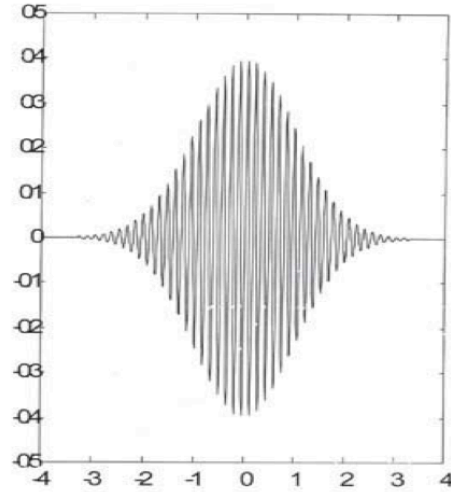
(c) Wigner base function is a harmonic wave modulated by a Gaussian envelope, which has infinite span but finite standard deviation for analyzing local dynamics. According to the uncertainty principle, the wave uncertainty reaches the minimum when its envelope is Gaussian.

Unfiltered & Filtered Gabor Distribution
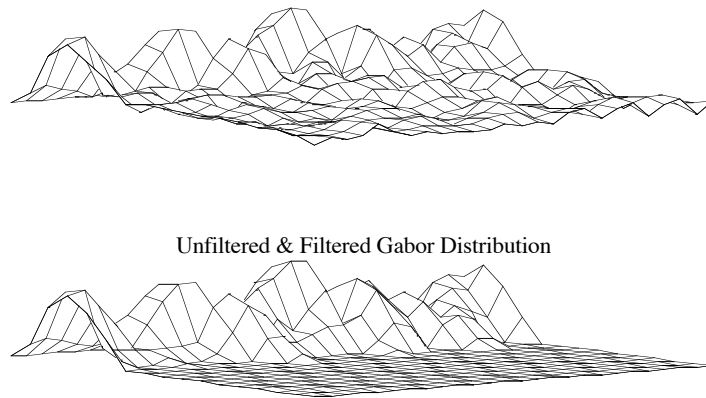
(d). The Gabor distribution for the unfiltered (upper figure) and filtered (lower figure) data of FSPCOMln HPc. The noise component (below sea level) was removed in two-dimensional Gabor space when the HPc series were decomposed by Wigner base functions, which minimize the uncertainty.

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Fig. 12. Construction and application of the time-variant filter in Gabor space.

We find strong evidence of persistent cycles and color chaos through HP cycles from financial and macro indexes. The best case is the S&P 500 index (its monthly data series are named as FSPCOM) in Fig. 13.

FSPCOM Raw HP Cycles
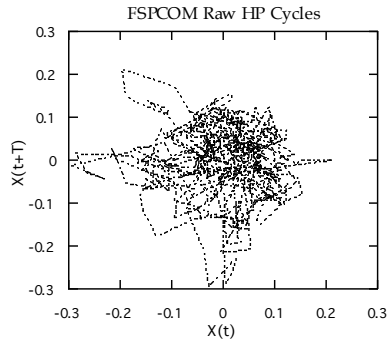
(a) Noisy image of unfiltered data

FSPCOM Filtered HP Cycles
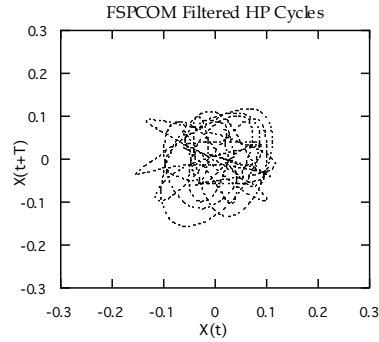
(b) Complex spiral of filtered data

FSPCOM Original & Filtered Cycles (H=0.5)

22

(c). The original $S_O(t)$ and filtered $S_g(t)$ from FSPCOM HPc.

The ratio of standard deviations of $S_g(t)$ over $S_O(t)$ series η=82.8%. Their correlation coefficient CCgo=0.847. The correlation dimension of the filtered series $S_O(t)$ is 2.5.

Fig. 13. Comparison of the original and filtered time series of FSPCOMln HP Cycles.

From Fig.13, we can see that the filtered $S_g(t)$ series explain 70% of variance the original $S_O(t)$ series, which is the cyclic component through the HP filter. Their cross-correlation is 0.847, which is very high in computational experiment. The correlation dimension is the numerical measurement of fractal dimension, which are 2.5 for S&P 500 monthly index. This is solid evidence that stock price movements can be characterized by nonlinear color chaos. Its average period is 3 years. Therefore, our analysis supports Schumpeter theory of business cycles [53]. He considered business cycles as biological clock, a living feature of economic organism.

IV. Why We Need a New Theoretical Framework for Economics and Finance?

The above discussions provide two kinds of mathematical representations. One is based on the linear-equilibrium framework for an effic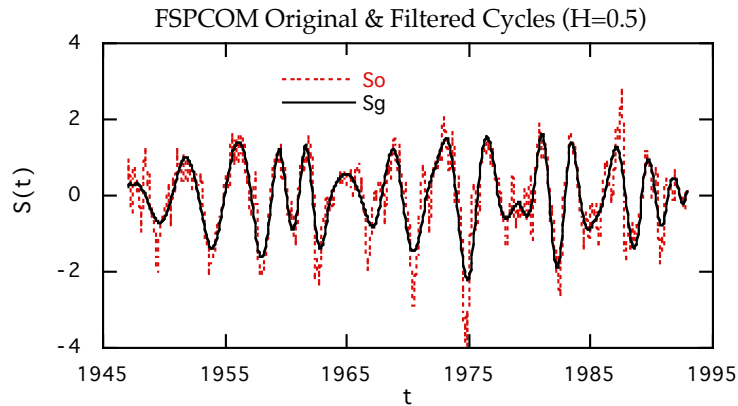ient market that is self-stabilizing without instability and crisis. Another is based on nonlinear non-equilibrium framework for economic complexity with cycles and chaos. Now the question is which framework is better for financial and economic research?

From mathematical perspective, linear representation is simpler and more beautiful, since linear problems often have analytical solutions, while nonlinear representation is more complex and hard to solve. Many nonlinear problems have no analytical solution and numerical solutions are often controversial. However, physicists prefer to use nonlinear math representation for three reasons: First, physicists have to deal with the real world that is nonlinear, non-equilibrium, non-stationary, and complex in nature. Second, the increasing computational power of computers is capable of solving nonlinear problems with desired precision. Third and the most important reason is that the new math framework based on nonlinear non-equilibrium approach open a new world in economics

and finance. Similar to non-Euclidean geometry in relativity theory, complexity science reveals new evidence of economic structure and historical changes, which is not known in linear equilibrium perspective.

In this section, we demonstrate that new discoveries in economic observation need new advanced math representation in empirical and theoretical analysis. Three lessons in 2008 financial crisis can be explained by our new approach based on the birth-death process, but not by equilibrium model based on geometric Brownian motion or dynamic stochastic general equilibrium (DSGE) model. First, 2008 financial crisis was originated in financial market started from derivative market collapse, which can be explained by our approach in terms of meso foundation of business fluctuations and breaking point in the master equation. Second, stock market bubbles often caused by animal behavior or herd action that can be described by social interaction in population dynamics, but not representative agent model in mainstream economics. Third, financial crisis implies sudden changes and regime switch that is beyond the scope of static distribution. In contrast, the birth-death process is a continuous-time model that can be solved by the master differential equation. Its probability distribution is varying over time that is compatible with historical observation. Its nonlinear solution is capable of characterizing phase transition and regime switch.

Let us see how advanced math representation opens new windows in economic analysis.

(4.1) The Principle of Large Numbers and the Discovery of Meso Foundation

Structure analysis plays an important role in physics and biology. However, structure is missing in macroeconomics. The so-called microfoundations theory simply asserts that macro dynamics should follow the same formulation in microeconomics. We will study the micro-macro relation in business cycle theory.

One fundamental issue in macro and finance theory is the origin of business cycles and the cause of the Great Depression. Lucas claimed that business cycles or even the Great Depression could be explained by workers' choices between work and leisure, which is called the micro-foundations theory of (macro) business cycles. How can we discover the micro-macro relation in stochastic mechanism? Schrödinger proposed a simple math that reveals the relation between the number of microelements and the degree of aggregate fluctuations [54]. We define the relative deviation (RD) as the ratio of the standard deviation to its mean when the underlying variable has only positive value, such as price and volume.

$$RD = \frac{STD(S_N)}{\sqrt{N}} \qquad\qquad (2)$$

Here, RD stands for relative deviation for positive variable, STD is standard deviation, which is the square root of the variance of a variable S with N elements: $S_N = X_1 + X_2 + ... + X_N$.

The idea is quite simple. The more element number N at the micro level, the less will be the aggregate fluctuation at the macro level, since independent fluctuations at the micro level would largely cancel out each other. We call this relation as the principle of large numbers. We extend this relation from static system to the population dynamics of the birth–death process [55]. We first calculate RD from an economic index through the HP filter. Then, we estimate the effective micro number N. The result is given in Table 2, which can be used for diagnosing financial crisis [56].

Table 2 Relative Deviation (RD) and Effective Number (N) for macro and finance Indexes

Item	RD (%)	N
Real personal consumption	0.15	800,000
Real GDP	0.2	500,000
Real private investment	1.2	10,000
Dow Jones Industrial (1928–2009)	1.4	9,000
S&P 500 Index (1947–2009)	1.6	5,000
NASDAQ (1971–2009)	2.0	3,000
Japan–US exchange rate (1971–2009)	6.1	300
US–Euro exchange rate (1999–2009)	4.9	400
Texas crude oil price (1978–2008)	5.3	400

In comparison, the number of households, corporations and public companies and the potential RD generated by them are given in Table 3.

Table 3 Numbers of Households and Firms in US (1980)

Micro-agents	Households	Corporations*	Public companies
N	80 700 000	2 900 000	20 000
RD (%)	0.01	0.1	0.7

* Here, we count only those corporations with more than $100 000 in assets.

From Tables 2 and 3, household fluctuations may contribute only about 5 percent of fluctuations in real gross domestic product (GDP) and less than 1 percent in real investment; and small firms can contribute 50 percent of fluctuations in real GDP or 8 percent in real investment. In contrast, public companies can generate about 60 percent of aggregate fluctuations in real investment. Clearly, there are very weak 'micro-foundations' but strong evidence of a 'meso-foundation' in macroeconomic fluctuations.

In another words, large macro fluctuations in macro and finance can only generated by fluctuations at the meso (finance) level, not the micro level from households or small firms. Extremely large fluctuations in commodity and currency market can only be caused by financial oligarchs. This is the root of 2008 financial crisis [57]. Therefore, competition policy is more effective than monetary and fiscal policy in stabilizing macro economy and financial market. Therefore, we strongly recommend that breaking-up financial oligarchs is the key to prevent the next financial crisis. This is the most important lesson that was learned from our new theoretical framework of the birth-death process.

Our approach finds strong evidence of meso (finance and industrial organization) structure from macro and finance indexes. Our three-level system of micro-meso-macro is better than the two-level system of micro and macro in Keynesian economics in studies of structural foundation of business cycles and crisis.

(4.2) Transition Probability in Calm and Turbulent Market: A Diagnosis of 2008 Financial Crisis

Efficient market theory in finance simply rules out the possibility of financial crisis. Static model of market instability is not capable in quantitative analysis of financial indexes. We apply non-equilibrium statistical mechanics to the birth-death process in analyzing the S&P 500 daily indexes from 1950 to 20010 [58]. The state-dependent transition probability is shown in Fig.14.
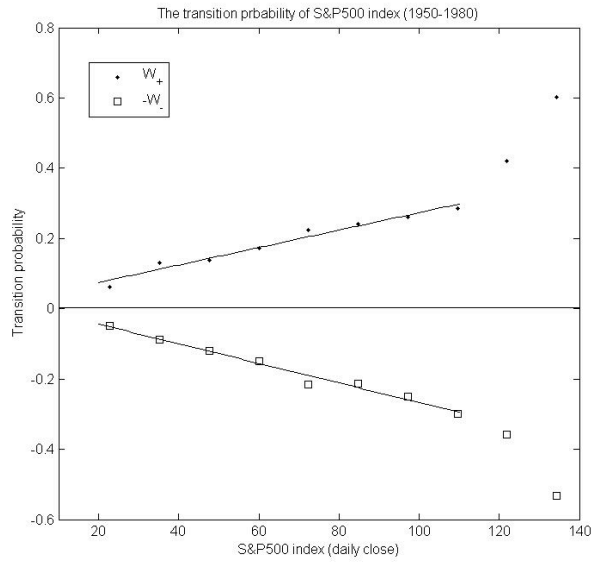
(a) Transition probability in (1950-1980).
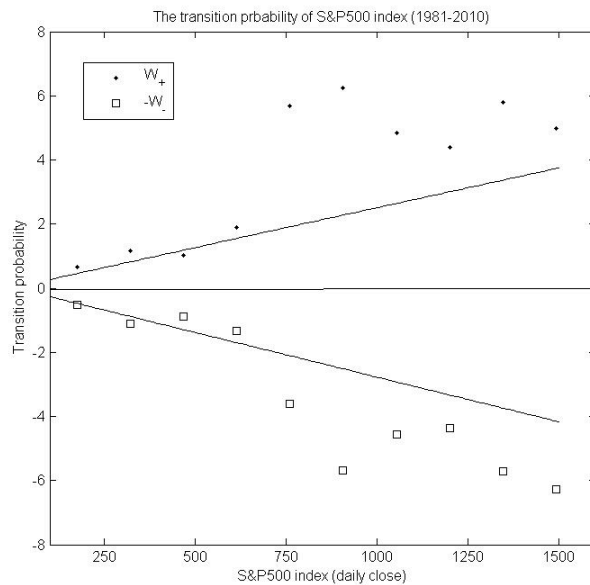
(b) Transition Probability in (1980-2010).

Fig. 14. Transition Probability for Calm (1950-1980) and Turbulent (1980-2010) Market Regimes.
The horizontal axis is the price level of the S&P 500 daily index. The vertical axis is the transition probability at varying price level. Data source is S&P 500 daily close prices.

From Fig.14, the upper curve can be explained by the "strength" with positive trading strategy, and the lower curve the strength with negative trading strategy. Intuitively, net price movements are resulted from the power balance between the "Bull camp" and the "Bear camp". There is remarkable difference between Period I (1950-1980) and Period II (1980-2010). Fig.14a is smoother than Fig.14b. The significant nonlinearity in Fig.14b is a visible sign of turbulent market that may produce financial crisis. Clearly, liberalization policy in Period II is closely related to the 2008 financial crisis in the sense that deregulation stimulated excess speculation in financial market.

We can solve the master equation of the birth-death process and find out the break point of the distribution probability. Our numerical solution indicates that the market breakdown occurs at the Sept. 25, 2008, when the Office of Thrift Supervision (OTS) seized Washington Mutual. This event was the peak from chain events preceding the 2008 financial crisis. The stock market went to panic since 26-Sep-2008. Our result is well compatible with historical timeline.

(4.3) Time-Varying High Moments and Crisis Warning in Market Dynamics

How to observe an evolving economy? If economy is time-varying in non-equilibrium situation, we can only make an approximation by local equilibrium through a moving time window. If we have many data points, we may choose a shorter time window to improve foresting in monitoring market instability. In the following example, we calculate statistical moments through a moving quarterly window for analyzing daily Dow Jones Industrial average (DJI) index that provides valuable information on coming crisis.

The sub-prime crisis in the U.S. reveals the limitation of diversification strategy based on mean-variance analysis. Neo-classical economics is mainly interested in the first two moments (mean and variance) for the whole series based on equilibrium perspective. For better understanding crisis dynamics, we exam high moments before, during, and after crisis in Fig.15 [59].
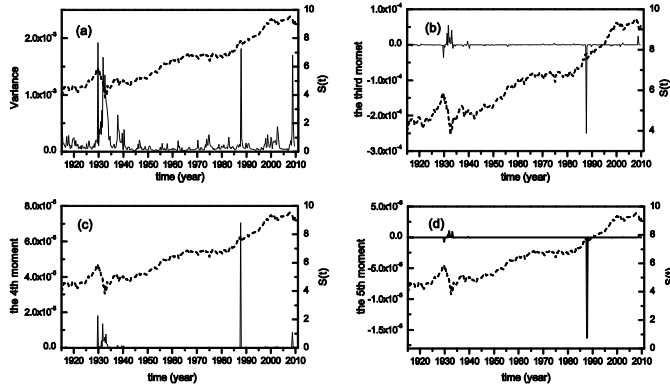
FIG. 15. The quarterly moments (solid lines) of the Dow-Jones Industrial Average (DJI) index. The original S(t) (dashed lines) is the natural logarithmic daily close price series. Each point in the solid line is calculated with a moving time window; its width is one quarter. Plots **a**, **b**, **c** and **d** correspond to $2^{nd,}$ 3rd, 4th and 5th moment, respectively. The magnitudes of each moment representation are 10^{-5} for variance, 10^{-8} for 3rd moment, 10^{-9} for 4th moment, and 10^{-11} for 5th moment. The daily data were from 2-Jan-1900 to 1-Sep-2010 with 27724 data points. Here we choose $\sigma_0^2 : 10^{-5}$ as the normal level. We would consider high moments when they reach the level of $10^{-1}\sigma_0^2$ or higher.

A regime switch and a turning point can be observed using a high moment representation and time-dependent transition probability. Financial instability is visible by dramatically increasing 3rd to 5th moments one-quarter before and during the crisis. The sudden rising high moments provide effective warning signals of a regime-switch or a coming crisis.

V. Math Selection and Philosophical Preference

Now, we have two systems of mathematical representations in economics and finance. Their main differences are given in Table 4.

Table 4. Competing ideas in economics and math

```
------------------------------------------------------------------------------
Subject  Math                        Economics
------------------------------------------------------------------------------
Linear-Equilibrium Paradigm (neoclassical)
------------------------------------------------------------------------------
         Single stable equilibrium    Self-stabilizing, laissez-fair
         Gaussian distribution        Efficient market, small deviation
         Representative agent         Rational man
         Optimization                 Complete market, closed sys.
         Exponential growth           No ecological constraints
                                      Consumerism, greedy

         White noise                  Short correlation, no history
         Brownian motion              Homogeneous behavior
         Regression analysis          Integrable system
         FD filter                    Short-term view, whitening signal
         Mean reversion               System convergence
         Mechanical philosophy        Reductionism
------------------------------------------------------------------------------
Nonlinear-Nonequilibrium Paradigm (Complex-Evolutionary Economics)
------------------------------------------------------------------------------
         Multiple equilibriums        Multiple regimes, instability, crisis
         Time-varying distribution    Calm & turbulent market
         Population dynamics          Social species within ecology
         Logistic wavelets            Tech. competition for resource
                                      Creative destruction
         Complex dynamics             Complex evolution in open system
         Color chaos & wavelets       Life cycle, biological clock,
                                      Medium-correlation
                                      History (path-dependency)
         Birth-death process          Herd behavior, social interaction
         Time-frequency analysis      Non-integrable, non-stationary sys.
         HP filter                    Medium-term view, trend-cycle
         Bifurcation tree             System divergence
         Biological philosophy        Holistic + structure = complexity
------------------------------------------------------------------------------
```

Neoclassical economics constructed a mathematical utopian of the market. The market is self-stabilizing without internal instability, since it has unique stable equilibrium. There is no resource limit to utility function and exponential growth. Therefore, human nature is greedy with unlimited want. Rational economic man makes independent decision without social interaction. The representative agent extremely simplifies economic math to a single-body problem, such as random walk, Brownian motion, and stochastic dynamic general equilibrium (SDGE) model. Resource allocation problem can be solved by optimization with complete market and perfect information without open competition and technology change. Price is the single variable that is capable of determining output and distribution. Business cycles and market fluctuations can be

explained by external shocks, so that there is little need for market regulation and government intervention. Reductionism and methodological individualism is justified by the linear approach of economic math since the whole equals the sum of parts.

The linear demand-supply curve is powerful in creating a market belief among public. That is why economic math is dominating in ivory tower economics. The only trouble for this linear-equilibrium world is a lacking understanding of recurrent crisis. One possible excuse is the existence of fat-tailed distribution. Even we have a big chance of large deviation, we still have no policy to deal with the bad luck, when we have a static unimodular probability distribution.

Complexity science and evolutionary economics discovered a complex world by nonlinear dynamics and Nonequilibrium physics. The real market has several regimes including calm and turbulent market, which can be described by multiple equilibrium and time-varying multi-modular distribution. Resource constraints shape nonlinear dynamics in economic decision and social interaction. Internal instability is driven by technology metabolism and social behavior. Human beings are social animals in nature. Macro and financial market must consider population dynamics and collective behavior. The representative agent model is misleading since collective behavior plays an important role in market fads and panic. Economic dynamics is complex since structure and history plays important role in economic movements. Life cycle in technology can be described by wavelets. Business cycles can be characterized by color chaos model of biological clock. Financial crisis can be better understood by time-frequency analysis and time-varying transition probability. Competition policy and leverage regulation provides new tools in market management in addition to monetary and fiscal policy.

Complexity science and evolutionary perspective is developing an alternative framework including micro, macro, finance, and institutional economics. Nonlinear math needs more understanding in history and social structure. The rapid development in big data and computer power paves the numerical foundation of the nonlinear-nonequilibrium economic math.

Now economists are facing a historical choice: how to select a proper math system for economics and finance.

There are three possible choices.

For strong believers in invisible hand and laissez-faire policy, existing neo-classical model is simple enough to portrait a self-stabilizing market, such as linear demand-supply curve, convex function in optimization, and noise-driven business cycle theory. The problem is that mainstream economics have to dodge difficult issues, such as excess volatility, herd behavior, and financial crisis.

For applied mathematicians in economic research, they often face a dilemma in choosing mathematical simplicity or computational

complexity. Linear models may have analytical solution that is beautiful for academic publication, while nonlinear models may be more realistic for policy practice but complicated for academic research. Currently, mainstream economic journals are falling behind science journals in advancing new math.

For empirical scientists doing economic research, there is an increasing trend in applying complexity science and non-equilibrium physics to economics and finance. New fields, such as the self-organization science, complex systems, and econophysics, are rapidly developed since 1980s.

Some economists may have doubt about physics application in economics. Our answer is: the proof of the pudding is in the eating. We already witness the success of physics application in medicine, including X-ray, ultrasound, laser, cardiograph, and DNA analysis. Economics more closely resembles medicine than theology. Complexity science and nonequilibrium physics has many applications in studying economics and finance, since the economy is an open system that is nonlinear and non-equilibrium in nature [60].

In philosophical perspective, the shift in math representation may facilitate a paradigm change [61]. The dominance of linear – equilibrium paradigm in mainstream economics sounds like theoretical theology in economics when it fails to address contemporary challenges, such as economic crisis and global warming [62]. Unlimited greed in human nature is not compatible to conservation law in physics and ecological constraints in earth [63].

The prevalence of noise-driven model in business cycle theory and regression analysis in econometrics is rooted in positive economics [64] or pragmatism in econometrics [65]. Now we have better knowledge why positive economics as well as econometrics fails to make economic prediction. Economic time series are non-stationary driven by structural changes in economy. There is little chance that model parameters may remain constant in time. That is why regression analysis is not reliable in economic research.

Mathematics is part of science knowledge that is constantly expanding by human experiences. Theoretical framework is developed from special knowledge to general knowledge, and accordingly, from simple to complex. For example, math developed from integer to fraction, from rational to irrational, from real to complex number, and from Euclidean to non-Euclidean geometry. Economic concepts will face similar changes in physics: from close to open systems, from equilibrium to non-equilibrium, and from simple to complex systems. Complex theory may integrate simple theory as its special case or an approximation at the lower order. Developing a new economic math is aimed to expanding our scope in economic thinking.

The fundamental debate between the exogenous and endogenous school in economics, or between the outside and inside view in philosophy, can be resolved by Prigogine's idea of "order through fluctuations"[66]. From the philosophical perspective of nonequilibrium physics, economies are open dissipative systems in nature. Structural evolution inside the system is strongly influenced by environmental changes outside the system. From mathematical perspective, all our new findings of color chaos and the birth-death process are observed through the HP filter. Its nonlinear smooth trend acts as a Copernicus reference system in analyzing macro and finance index. Its economic implication is replacing rational expectations by market trends that are shaped by reflexivity [67], or social interactions between global environment and market players. In our new option-pricing model based on collective behavior, the constant risk-free interest rate is replacing by the trend price in our model based by the birth-death process [68].

There is always a trade-off between mathematical simplicity and empirical relevance. We choose the birth-death process because it is the simplest model in population dynamics that is solvable by master equation. Although the birth-death process is more complex than the Brownian motion model of the representative agent, but we obtain more knowledge of social psychology. By the similar token, we learn more about changing economies from time-frequency analysis than that from static statistics. We take the experimental approach in choosing proper level of computational complexity. There are so many innovations in computational algorithms and data mining.

We found that theoretical physics including nonlinear dynamics, quantum mechanics and statistical mechanics have more explaining power in economic analysis for three reasons.

First, theoretical physics has a general and consistent framework in theory, while other numerical algorithms only have technical merits without deep theoretical foundation. Only physics theory provides a unified framework for physics, chemistry, biology, medicine, and now economics.

Second, neoclassical economics chooses the wrong framework from physics. Classical mechanics started from Newton's law in mechanics that is universal when particle speed was far below light speed. Hamiltonian formulation of classical mechanics is valid only for conservative system without energy dissipation. Neoclassical economics borrowed the Hamiltonian formulation to justify utility and profit maximization in a utopian world without energy dissipation and technology innovation. Non-equilibrium physics realized the fundamental differences between conservative system with time symmetry and dissipative system with time asymmetry. Living and social systems are dissipative systems in nature, which are evolving in time and space. Therefore, non-equilibrium forces,

such as unpredictable uncertainty [69] and creative destruction [70], have more weight than equilibrium forces, such as cost competition and arbitrage activity, in driving economic movements. Newton mechanics did not exclude nonlinear interaction. Physicists discovered deterministic chaos from both conservative system and dissipative system. However, the requirement of convex set in microeconomics simply rules out of possibility of multiple equilibrium and chaos. We should reconstruct theoretical foundation of microeconomics, so that it is compatible with observed patterns in macro and finance.

Third, empirical evidence of economic complexity and economic chaos impose severe limits to economic reductionism and methodological individualism. Nonlinear interaction and multi-layer structure uncover the importance of evolutionary perspective and holistic view of human activity. From evolutionary perspective, nonlinearity and non-equilibrium mechanism is mathematical representation of irreversibility and history. Complex patterns from mathematical representation are hard to understand if we do not have relevant knowledge in economic structure and social history. In this sense, nonequilibrium and nonlinear approach may serve as a bridge to the Two Cultures [71]. We learn a great deal from wide-range dialogue with biologists, historians, sociologists, anthropologists, and psychologists, in addition to economists from different schools.

VI. Conclusion

In philosophy, economists always believe that economic behavior should be more complex than those in physics. In practice, economic math is much simpler than math in physics. Typical example is the ideal gas model, which is simplest in statistical mechanics. Standard statistical mechanics started with an ensemble with many particles, whose distribution is determined by the underlying dynamics. In contrast, macro and finance theory in economics only have one particle in the representative agent model with fixed probability distribution [72]. Therefore, economic theory is more naïve than physics in mathematical representation. We need judge the economic philosophy by their deeds rather than words.

Historically, mathematical representation goes hand in hand with technology and social changes. Economic math also undergoes rapid development from simple to more advanced stage. Mathematical representation plays an important role in economic theory and empirical analysis. Economic math models are competing in three aspects: empirical explaining power, mathematical elegance, and philosophical implications.

From the historical lessons in science, it is empirical relevance of mathematical model that defines the meaning of mathematical beauty and philosophical status. For example, Einstein's relativity theory has better empirical foundation than Newton's mechanics. The success of relativity theory changed mathematical status of non-Euclidean geometry and philosophical foundation of absolute universe. In economics, natural experiments, such as the Great Depression and the 2008 Financial Crisis, shake the belief in the self-stabilizing market. People demand new thinking in economics. The nonlinear and non-equilibrium nature of the modern economy will be finally accepted by the public and economics community.

There are three forces that may accelerate changes in economic thinking.

First, big data and increasing computer power will stimulate upgrading in mathematical representation from linear, static models to nonlinear, non-stationary models in economic theory.

Second, tremendous progress of complexity science has made increasing application in science, engineering, and medicine. Their successes will spread to economics and finance. The closed atmosphere in mainstream economics cannot be last long.

Third, neoclassical economics is mainly the product of Anglo-Saxon culture based on individualism. The 2008 financial crisis and global warming marked the end of consumerism. Both developed countries and developing countries face severe challenges from environment, resource, and poverty. Economic development has to adapt to diversified environmental conditions. This social awareness will speed up paradigm changes in economic thinking.

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